Chapter 9
Online Algorithms

Algorithm Theory
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Online Computations

- Sometimes, an algorithm has to start processing the input before the complete input is known

- For example, when storing data in a data structure, the sequence of operations on the data structure is not known

Online Algorithm: An algorithm that has to produce the output step-by-step when new parts of the input become available.

Offline Algorithm: An algorithm that has access to the whole input before computing the output.

- Some problems are inherently online
  - Especially when real-time requests have to be processed over a significant period of time

Competitive Ratio

- Let's again consider optimization problems
  - For simplicity, assume, we have a minimization problem

Optimal offline solution $\mathbf{OPT}(\boldsymbol{I})$:

- Best objective value that an offline algorithm can achieve for a given input sequence I

Online solution $\mathbf{ALG}(\boldsymbol{I})$:

- Objective value achieved by an online algorithm ALG on I

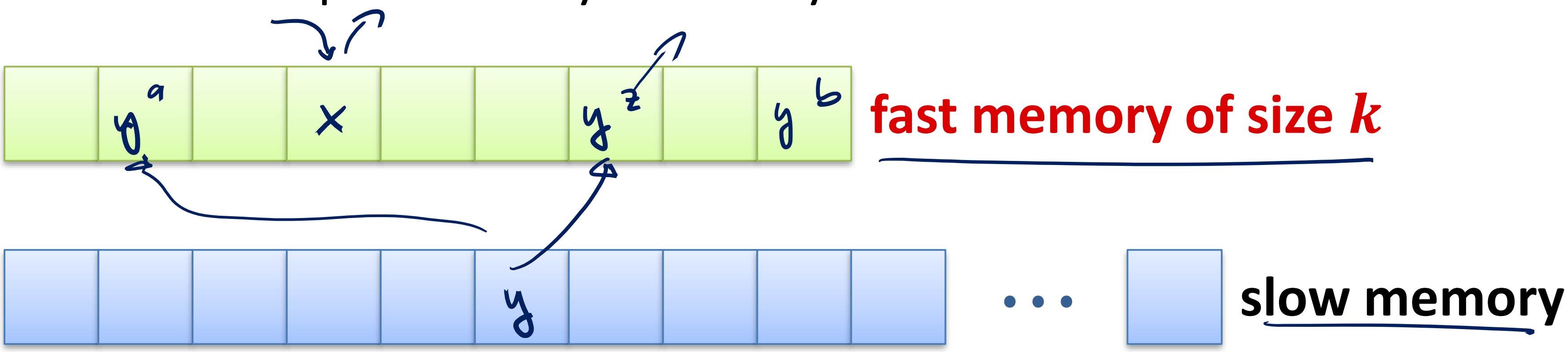
Competitive Ratio: An algorithm has competitive ratio $c \geq 1$ if

$$\mathbf{ALG}(\boldsymbol{I}) \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha = 0$, we say that ALG is strictly c-competitive.

Paging Algorithm

Assume a simple memory hierarchy:



If a memory page has to be accessed:

- Page in fast memory (hit): take page from there
- Page not in fast memory (miss): leads to a page fault
- Page fault: the page is loaded into the fast memory and some page has to be evicted from the fast memory
- Paging algorithm: decides which page to evict
- Classical online problem: we don't know the future accesses

Paging Strategies

Least Recently Used (LRU):

- Replace the page that hasn't been used for the longest time

First In First Out (FIFO):

- Replace the page that has been in the fast memory longest

Last In First Out (LIFO):

- Replace the page most recently moved to fast memory

Least Frequently Used (LFU):

- Replace the page that has been used the least

Longest Forward Distance (LFD): optimal offline algorithm

- Replace the page whose next request is latest (in the future)
- LFD is **not** an online strategy!

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

- For contradiction, assume that LFD is not optimal
- Then there exists a finite input sequence σ on which LFD is not optimal (assume that the length of σ is $|\sigma| = n$)
- Let OPT be an optimal solution for σ such that
  - OPT processes requests $1, \dots, i$ in exactly the same way as LFD
  - OPT processes request $i + 1$ differently than LFD
  - Any other optimal strategy processes one of the first $i + 1$ requests differently than LFD
- Hence, OPT is the optimal solution that behaves in the same way as LFD for as long as possible → we have $i < n$
- Goal: Construct OPT' that is identical with LFD for req. $1, \dots, i + 1$

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 1: Request $i + 1$ does **not** lead to a page fault

- LFD does not change the content of the fast memory
- OPT behaves differently than LFD
  → OPT replaces some page in the fast memory
  - As up to request $i + 1$, both algorithms behave in the same way, they also have the same fast memory content
  - OPT therefore does not require the new page for request $i + 1$
  - Hence, OPT can also load that page later (without extra cost) → OPT′

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 2: Request $i + 1$ does lead to a **page fault**

- LFD and OPT move the same page into the fast memory, but they evict different pages
  - If OPT loads more than one page, all pages that are not required for request $i + 1$ can also be loaded later
- Say, LFD evicts page p and OPT evicts page p'
- By the definition of LFD, p' is required again before page p

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

$j > j'$

Case 2: Request $i+1$ does lead to a **page fault**

$i+1$ p'/p

$\ell' < \ell$: OPT evicts p p/p'

j': next req. for p'

j: next req. for p

LFD evicts p

OPT evicts p'

keeps p

$\ell \leq j'$: OPT loads p' (for first time after $i+1$)

a) OPT keeps p in fast memory until request ℓ
   - Evict p at request $i+1$, keep p' instead and load p (instead of p') back into the fast memory at request ℓ

b) OPT evicts p at request $\ell' < \ell$
   - Evict p at request $i+1$ and p' at request ℓ' (switch evictions of p and p')