Exercise 1: Decidability

(a) Is the following language decidable? Either prove that it is not decidable or provide an algorithm that decides it.

\[ \text{INDEPENDENT SET} = \{ \langle G, k \rangle \mid G \text{ is a graph and contains an independent set of size } k \} \]

Remark: An independent set of a graph with size \( s \) is a set \( S \subseteq V, |S| = s \) such that \( \{v, w\} \notin E \) for all \( u, w \in S \) and \( |S| \) is its size.

(b) Let \( H \) be the language of the halting problem. Give a language \( L \) such that \( L \cap H \) decidable and give a language \( K \) such that \( K \cap H \) is undecidable. Prove your claims.

Exercise 2: \( \mathcal{O} \)-Notation Formal Proofs

The set \( \mathcal{O}(f) \) contains all functions that are asymptotically not growing faster than the function \( f \) (when additive or multiplicative constants are neglected). That is:

\[ g \in \mathcal{O}(f) \iff \exists c \geq 0, \exists M \in \mathbb{N}, \forall n \geq M : g(n) \leq c \cdot f(n) \]

For the following pairs of functions, check whether \( f \in \mathcal{O}(g) \) or \( g \in \mathcal{O}(f) \) or both. Proof your claims (you do not have to prove a negative result \( \not\in \), though).

(a) \( f(n) = 100n \), \( g(n) = 0.1 \cdot n^2 \)

(b) \( f(n) = \log_2(n!) \), \( g(n) = n \log_2 n \)

(c) \( f(n) = 2^n \), \( g(n) = 3^n \)

[Hint: \( n! := \prod_{i=1}^{n} i \geq (n/2)^{n/2} \)]

Remark: It is easy to produce tons of exercises of this type. Create a few exercises and try to solve them to practice this for the exam!

Exercise 3: Sort Functions by Asymptotic Growth

Sort the following functions by asymptotic growth using the \( \mathcal{O} \)-notation. Write \( g \prec_\mathcal{O} f \) if \( g \in \mathcal{O}(f) \) and \( f \notin \mathcal{O}(g) \). Write \( g \equiv_\mathcal{O} f \) if \( f \in \mathcal{O}(g) \) and \( g \in \mathcal{O}(f) \).

\[
\begin{array}{cccc}
n^2 & \sqrt{n} & 2^n & \log(n^2) \\
3^n & n^{100} & \log(\sqrt{n}) & (\log n)^2 \\
\log n & 10^{100}n & n! & n \log n \\
n \cdot 2^n & n^n & \sqrt{\log n} & n \\
\end{array}
\]