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Theoretical Computer Science - Bridging Course
Winter Term 2018
Exercise Sheet 10

for getting feedback submit (electronically) before the start of the tutorial on 14th of January 2019.

Exercise 1: Propositional Logic: Basic Terms   *(2+2+2+2 Points)*

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \wedge \neg q) \vee (\neg p \vee q)$

(b) $\varphi_2 = (\neg p \wedge (\neg p \vee q)) \leftrightarrow (p \vee \neg q)$

(c) $\varphi_3 = (p \wedge \neg q) \to \neg(p \wedge q)$

(d) $\varphi_4 = (p \wedge q) \to (p \vee r)$

Remark: $a \to b :\equiv \neg a \vee b$, $a \leftrightarrow b :\equiv (a \to b) \wedge (b \to a), a \not\to b :\equiv \neg(a \to b)$.

Exercise 2: CNF and DNF   *(2+2 Points)*

(a) Convert
$$\psi_1 := (x \wedge y \to z \vee w) \wedge (y \to x) \wedge (z \wedge y \to 0) \wedge (w \wedge y \to 0) \wedge y$$
into Conjunctive Normal Form (CNF).

(b) Convert
$$\psi_2 := \neg((\neg p \leftrightarrow \neg q) \wedge (\neg r \to q))$$
into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step. Note that 2a is not ambigious as there are clear rules for interpreting such a formula without additional parentheses.

Exercise 3: Logical Entailment   *(2+2 Points)*

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \vee (q \wedge \neg r), \neg r \wedge p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b) $\varphi_2 := (q \leftrightarrow r) \to p$

Exercise 4: Inference Rules and Calculi   *(2+2 Points)*

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* $\mathbf{C}$ is described by a *set* of inference rules.

Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus $\mathbf{C}$ to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{(\varphi \wedge \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \wedge q, p \to r, (q \wedge r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.