Theoretical Computer Science - Bridging Course  
Winter Term 2018  
Exercise Sheet 11

for getting feedback submit (electronically) before the start of the tutorial on  
This is the last exercise sheet!

Exercise 1: Understanding FO Logic  
(3+2+3 Points)

Consider the following first order logical formulae

\[
\varphi_1 := \forall x R(x, x)  
\]

\[
\varphi_2 := \forall x \forall y R(x, y) \rightarrow (\exists z R(x, z) \land R(z, y))  
\]

\[
\varphi_3 := \exists x \exists y (\neg R(x, y) \land \neg R(y, x))  
\]

where \( x, y \) are variable symbols and \( R \) is a binary predicate. Give an interpretation

(i) \( I_1 \) which is a model of \( \varphi_1 \land \varphi_2 \).

(ii) \( I_2 \) which is no model of \( \varphi_1 \land \varphi_2 \land \varphi_3 \).

(iii) \( I_3 \) which is a model of \( \varphi_1 \land \varphi_2 \land \varphi_3 \).

Exercise 2: Truth Value  
(6 Points)

Determine the truth value of the statement \( \exists x \forall y (x \leq y^2) \) if the domain (or universe) for the variables consists of:

(a) the positive real numbers,

(b) the integers,

(c) the nonzero real numbers.
Exercise 3: Resolution Calculus  \hspace{1cm} (2+4 \text{ Points})

Due to the Contradiction Theorem (cf. lecture) for every knowledge base $KB$ and formula $\varphi$ it holds

$$KB \models \varphi \iff KB \cup \{\neg \varphi\} \not\models \bot.$$  

Remark: $\bot$ is a formula that is unsatisfiable.

Thus, in order to show that $KB$ entails $\varphi$, we show that $KB \cup \{\neg \varphi\}$ entails a contradiction. A calculus $C$ is called refutation-complete if for every knowledge base $KB$

$$KB \models \bot \implies KB \vdash_C \bot.$$  

Therefore, if we have a refutation-complete calculus $C$, it suffices to show $KB \cup \{\neg \varphi\} \vdash_C \bot$ in order to prove $KB \models \varphi$.

The Resolution Calculus$^1$ $R$ is correct and refutation-complete for knowledge bases that are given in Conjunctive Normal Form (CNF). A knowledge base $KB$ is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of $m_i$ literals $L_{i,j}$

Remark: $KB$ represents the formula $C_1 \land \ldots \land C_n$ with $C_i = L_{i,1} \lor \ldots \lor L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the resolution rule:

$$R : \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$  

Remark: $L$ is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in $KB$ ($C_1, C_2$ may be empty). To show $KB \vdash_R \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses $L$ and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L, \neg L\} \vdash_R \bot$).

Consider the following propositional formula

$$\psi := (x \land y \to z \lor w) \land (y \to x) \land (z \land y \to 0) \land (w \land y \to 0) \land y.$$  

Use the resolution calculus to show that $\psi$ is unsatisfiable.

Remark: You first have to convert $\psi$ into CNF which you already should have done in one of the previous exercises.

Remark: The ’net’ is full of similar exercises. Practice them for the exam!

\hspace{10cm}$^1$Complete calculi are impractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.