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Theoretical Computer Science - Bridging Course
Summer Term 2018
Exercise Sheet 1

for getting feedback submit (electronically) before the start of the tutorial on 29th of October 2018.

Exercise 1: Induction *(7 Points)*

Find a much more compact formula for the term $\sum_{k=1}^{n}(2k-1)$ and prove its correctness by induction.
Hint: $\frac{n(n+1)}{2}$ *would be such a formula for the expression* $\sum_{k=1}^{n} k$.

Sample Solution

We claim that $\sum_{k=1}^{n}(2k-1)$ equals n^2 and show the claim by induction.

Induction Start: For $n = 1$ the claim follows as $\sum_{k=1}^{1}(2k-1) = 2-1 = 1 = 1^2$.

Induction Hypothesis/Step: Now assume the statement is true for spme n. It follows that

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + (2(n+1)-1) && (1)\\
&= n^2 + (2(n+1)-1) && (2)\\
&= n^2 + 2n + 1 && (3)\\
&= (n+1)^2\ , && (4)
\end{aligned}$$

which shows that the statement also holds for $n+1$. The second equality in the equation above comes from the assumption for n.
Thus the claim follows with the principle of induction.

Exercise 2: Even Number of Odd Degree Nodes *(5 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ of an undirected graph $G = (V, E)$ is the number of its neighbors, i.e,

$$d(v) = |\{u \in V \mid \{v, u\} \in E\}|.$$

Show that the number of nodes with odd degree in every simple graph is even.

Hint: Consider the sum $D = \sum_{v \in V} d(v)$ *of all degrees. Is* D *odd or even?*

Sample Solution

Let $G = (V, E)$ be a simple graph. Every edge contributes 2 to D, hence $D = 2|E|$. Therefore, D is an even number. Let V_e (V_d) be the vertices with even (odd) degree, respectively.
Then we obtain that $D = \sum_{v \in V} d(v) = \sum_{v \in V_e} d(v) + \sum_{v \in V_o} d(v)$. Now, subtract $\sum_{v \in V_e} d(v)$ from both sides. We obtain that $\sum_{v \in V_d} d(v) = D - \sum_{v \in V_e} d(v)$ is even because the right hand side is the subtraction of two even numbers. The left hand side is a sum of odd numbers and to be even there has to be an even number of summands, i.e., $|V_o|$ is even.

Exercise 3: Playing with Sets *(8 Points)*

Let A be a set. Show that the following three statements are equivalent.

(i) $B \setminus A = B$ for all sets B,

(ii) $(A \cup B) \setminus A = B$ for all sets B,

(iii) $A = \emptyset$.

Hint: It is sufficient to prove that $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ *and* $(iii) \Rightarrow (i)$.

Sample Solution

$(i) \Rightarrow (ii)$. Let B be some set. We show both inclusions ($(A \cup B) \setminus A \subseteq B$ and $(A \cup B) \setminus A \supseteq B$) separately. s $' \subseteq'$: Let $x \in A \cup B \setminus A$, that is, $x \in A \cup B$ and $x \not A$. Hence $x \in B$. (we did not use the assumptions in (i) to show this).
$' \supseteq'$: Assume that there is some $x \in B$ that is not contained in $(A \cup B) \setminus A$. That is $x \in A$. This is a contradiction to (i) as $B \setminus A \neq B$.
$(ii) \Rightarrow (iii)$ Assume that $A \neq \emptyset$, that is, there is some $x \in A$. Let $B = \{x\} \subseteq A$. Then $(A \cup B) \setminus A = (A) \setminus A = \emptyset \neq B$. The claim holds.
$(iii) \Rightarrow (i)$ Let B be a set. Then $B \setminus A \subseteq B$ holds. For the reverse inclusion let $x \in B$, as $A = \emptyset$ we have $x \neq A$ and we obtain $x \in B \setminus A$.