Exercise 1: Induction

Find a much more compact formula for the term \( \sum_{k=1}^{n} (2k - 1) \) and prove its correctness by induction.

\( \text{Hint: } \frac{n(n+1)}{2} \text{ would be such a formula for the expression } \sum_{k=1}^{n} k. \)

Sample Solution

We claim that \( \sum_{k=1}^{n} (2k - 1) \) equals \( n^2 \) and show the claim by induction.

\textbf{Induction Start: } For \( n = 1 \) the claim follows as \( \sum_{k=1}^{1} (2k - 1) = 2 - 1 = 1 = 1^2. \)

\textbf{Induction Hypothesis/Step: } Now assume the statement is true for some \( n \). It follows that

\[
\begin{align*}
\sum_{k=1}^{n+1} (2k - 1) &= \sum_{k=1}^{n} (2k - 1) + (2(n + 1) - 1) \\
&= n^2 + (2(n + 1) - 1) \\
&= n^2 + 2n + 1 \\
&= (n + 1)^2,
\end{align*}
\]

which shows that the statement also holds for \( n + 1 \). The second equality in the equation above comes from the assumption for \( n \).

Thus the claim follows with the principle of induction.

Exercise 2: Even Number of Odd Degree Nodes

A \textit{simple graph} is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree \( d(v) \) of a node \( v \in V \) of an undirected graph \( G = (V, E) \) is the number of its neighbors, i.e,

\[ d(v) = |\{u \in V \mid \{v, u\} \in E\}|. \]

Show that the number of nodes with odd degree in every simple graph is even.

\( \text{Hint: } \text{Consider the sum } D = \sum_{v \in V} d(v) \text{ of all degrees. Is } D \text{ odd or even?} \)
Sample Solution

Let \( G = (V, E) \) be a simple graph. Every edge contributes 2 to \( D \), hence \( D = 2|E| \). Therefore, \( D \) is an even number. Let \( V_e \) (\( V_o \)) be the vertices with even (odd) degree, respectively.

Then we obtain that \( D = \sum_{v \in V} d(v) = \sum_{v \in V_e} d(v) + \sum_{v \in V_o} d(v) \). Now, subtract \( \sum_{v \in V_e} d(v) \) from both sides. We obtain that \( \sum_{v \in V_d} d(v) = D - \sum_{v \in V_o} d(v) \) is even because the right hand side is the subtraction of two even numbers. The left hand side is a sum of odd numbers and to be even there has to be an even number of summands, i.e., \( |V_o| \) is even.

Exercise 3: Playing with Sets

Let \( A \) be a set. Show that the following three statements are equivalent.

(i) \( B \setminus A = B \) for all sets \( B \),

(ii) \( (A \cup B) \setminus A = B \) for all sets \( B \),

(iii) \( A = \emptyset \).

Hint: It is sufficient to prove that (i) \(\Rightarrow\) (ii), (ii) \(\Rightarrow\) (iii) and (iii) \(\Rightarrow\) (i).

Sample Solution

(i) \(\Rightarrow\) (ii). Let \( B \) be some set. We show both inclusions \( (A \cup B) \setminus A \subseteq B \text{ and } (A \cup B) \setminus A \supseteq B \) separately. \( s' \subseteq' : \) Let \( x \in A \cup B \setminus A \), that is, \( x \in A \cup B \) and \( x \notin A \). Hence \( x \in B \). (we did not use the assumptions in (i) to show this).

\( s' \supseteq' : \) Assume that there is some \( x \in B \) that is not contained in \( (A \cup B) \setminus A \). That is \( x \in A \). This is a contradiction to (i) as \( B \setminus A \neq B \).

(ii) \(\Rightarrow\) (iii) Assume that \( A \neq \emptyset \), that is, there is some \( x \in A \). Let \( B = \{x\} \subseteq A \). Then \( (A \cup B) \setminus A = (A) \setminus A = \emptyset \neq B \). The claim holds.

(iii) \(\Rightarrow\) (i) Let \( B \) be a set. Then \( B \setminus A \subseteq B \) holds. For the reverse inclusion let \( x \in B \), as \( A = \emptyset \) we have \( x \neq A \) and we obtain \( x \in B \setminus A \).