Exercise 1: Decidability  \( (5+5 \text{ Points}) \)

(a) Is the following language decidable? Either prove that it is not decidable or provide an algorithm that decides it.

**INDEPENDENT SET** = \{⟨\(G, k\)⟩ | \(G\) is a graph and contains an independent set of size \(k\)\}.

*Remark:* An independent set of a graph with size \(s\) is a set \(S \subseteq V\), \(|S| = s\) such that \({v, w}\) \(\notin E\) for all \(u, w \in S\) and \(|S|\) is its size.

(b) Let \(H\) be the language of the halting problem. Give a language \(L\) such that \(L \cap H\) decidable and give a language \(K\) such that \(K \cap H\) is undecidable. Prove your claims.

**Sample Solution**

(a) We show the result by giving an algorithm. First we check whether the input is an encoding of a tuple \(G, k\). If so, we assume that the graph is stored with an adjacency matrix. Then we simply iterate through all \(k\)-tuples \(v_1, \ldots, v_k\) of nodes in \(V\) and for each of them we test whether none of the edges \({v_i, v_j}\), \(i \neq j\) exist. If this is the case for any of the \(k\) tuples we return true, otherwise false.

(b) \(L = \emptyset\) and \(K = H\).

Exercise 2: \(\mathcal{O}\)-Notation Formal Proofs  \( (2+2+2 \text{ Points}) \)

The set \(\mathcal{O}(f)\) contains all functions that are asymptotically not growing faster than the function \(f\) (when additive or multiplicative constants are neglected). That is:

\[ g \in \mathcal{O}(f) \iff \exists c \geq 0, \exists M \in \mathbb{N}, \forall n \geq M : g(n) \leq c \cdot f(n) \]

For the following pairs of functions, check whether \(f \in \mathcal{O}(g)\) or \(g \in \mathcal{O}(f)\) or both. Proof your claims (you do not have to prove a negative result \(\notin\), though).

(a) \(f(n) = 100n, \ g(n) = 0.1 \cdot n^2\)

(b) \(f(n) = \log_2(n!), \ g(n) = n \log_2 n\)  \[\text{[Hint: } n! := \prod_{i=1}^{n} i \geq (n/2)^{n/2}]\]

(c) \(f(n) = 2^n, \ g(n) = 3^n\)

Remark: It is easy to produce tons of exercises of this type. Create a few exercises and try to solve them to practice this for the exam!
Sample Solution

(a) It is $100n \in O(0.1n^2)$. To show that we require constants $c, M$ such that $100n \leq c \cdot 0.1n^2$ for all $n \geq M$. Obviously this is the case for $c = 1000$ and $M = 1$.

(b) We have

$$\log_2(n!) \leq \log_2(n^n) = n \log_2 n$$

for all $n \geq 1$. Therefore $\log_2(n!) \in O(n \log_2 n)$. In the other direction we have the following result

$$\log_2(n!) \geq \log_2((n/2)^{n/2}) = \frac{n}{2} \log_2(n/2) = \frac{n}{2} \log_2 n - \frac{n}{2} \geq \frac{n}{2} \log_2 n - \frac{n}{4} \log_2 n = \frac{1}{4} n \log_2 n$$

For all $n \geq 4$. Thus $n \log_2 n \in O(\log_2(n!))$ is also the case. In general if both $f \in O(g)$ and $g \in O(f)$ are true, these two functions are called asymptotically equivalent in terms of the $O$-notation. This is denoted by $f \in \Theta(g)$.

(c) Obviously $2^n \leq 3^n$ for all $n \geq 1$. The converse is false though, because a $c$ such that $3^n \leq c2^n$ must fulfill $c \geq (3/2)^n$ for arbitrarily big $n$, but since $(3/2)^n$ is unbounded there can be no such $c$.

Exercise 3: Sort Functions by Asymptotic Growth  
(6 Points)

Sort the following functions by asymptotic growth using the $O$-notation. Write $g \prec O f$ if $g \in O(f)$ and $f \notin O(g)$. Write $g = O f$ if $f \in O(g)$ and $g \in O(f)$.

<table>
<thead>
<tr>
<th>$n^2$</th>
<th>$\sqrt{n}$</th>
<th>$2^n$</th>
<th>$\log(n^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^n$</td>
<td>$n^{100}$</td>
<td>$\log(\sqrt{n})$</td>
<td>$(\log n)^2$</td>
</tr>
<tr>
<td>$\log n$</td>
<td>$10^{100}n$</td>
<td>$n!$</td>
<td>$n \log n$</td>
</tr>
<tr>
<td>$n \cdot 2^n$</td>
<td>$n^n$</td>
<td>$\sqrt{\log n}$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Sample Solution

$$< O \quad \sqrt{\log n} \quad < O \quad \log(\sqrt{n}) = O \quad \log n = O \quad \log(n^2)$$

$$< O \quad (\log n)^2 \quad < O \quad \sqrt{n} \quad < O \quad n \quad = O \quad 10^{100}n$$

$$< O \quad n \log n \quad < O \quad n^2 \quad < O \quad n^{100} \quad < O \quad 2^n$$

$$< O \quad n \cdot 2^n \quad < O \quad 3^n \quad < O \quad n! \quad < O \quad n^n$$