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Theoretical Computer Science - Bridging Course
Summer Term 2018
Exercise Sheet 7

for getting feedback submit (electronically) before the start of the tutorial on 10th of December 2018.

Exercise 1: Decidability *(5+5 Points)*

(a) Is the following language decidable? Either prove that it is not decidable or provide an algorithm that decides it.

Independent Set $=\{\langle G, k\rangle \mid G$ is a graph and contains an independent set of size $k\}$.

Remark: An independent set of a graph with size s is a set $S \subseteq V$, $|S| = s$ such that $\{v, w\} \notin E$ for all $u, w \in S$ and $|S|$ is its size..

(b) Let H be the language of the halting problem. Give a language L such that $L \cap H$ decidable and give a language K such that $K \cap H$ is undecidable. Prove your claims.

Sample Solution

(a) We show the result by giving an algorithm. First we check whether the input is an encoding of a tuple G, k. If so, we assume that the graph is stored with an adjacency matrix. Then we simply iterate through all k-tuples $v_1, \ldots, v_k$ of nodes in V and for each of them we test whether none of the edges $\{v_i, v_j\}, i \neq j$ exist. If this is the case for any of the k tuples we return true, otherwise false.

(b) $L = \emptyset$ and $K = H$.

Exercise 2: $\mathcal{O}$-Notation Formal Proofs *(2+2+2 Points)*

The set $\mathcal{O}(f)$ contains all functions that are asymptotically not growing faster than the function f (when additive or multiplicative constants are neglected). That is:

$$g \in \mathcal{O}(f) \Longleftrightarrow \exists c \geq 0, \exists M \in \mathbb{N}, \forall n \geq M : g(n) \leq c \cdot f(n)$$

For the following pairs of functions, check whether $f \in \mathcal{O}(g)$ or $g \in \mathcal{O}(f)$ or both. Proof your claims (you do not have to prove a negative result $\notin$, though).

(a) $f(n) = 100n,\ g(n) = 0.1 \cdot n^2$

(b) $f(n) = \log_2(n!),\ g(n) = n \log_2 n$ [*Hint:* $n! := \prod_{i=1}^{n} i \geq (n/2)^{n/2}$]

(c) $f(n) = 2^n,\ g(n) = 3^n$

Rermark: It is easy to produce tons of exercises of this type. Create a few exercises and try to solve them to practice this for the exam!

Sample Solution

(a) It is $100n \in \mathcal{O}(0.1n^2)$. To show that we require constants c, M such that $100n \leq c \cdot 0.1n^2$ for all $n \geq M$. Obviously this is the case for $c = 1000$ and $M = 1$.

(b) We have

$$\log_2(n!) \leq \log_2(n^n) = n \log_2 n$$

for all $n \geq 1$. Therefore $\log_2(n!) \in \mathcal{O}(n \log_2 n)$. In the other direction we have the following result

$$\log_2(n!) \overset{\text{Hint}}{\geq} \log_2((n/2)^{\frac{n}{2}}) = \frac{n}{2}(\log_2 n - \log_2 2) = \frac{n}{2}\log_2 n - \frac{n}{2} \overset{\text{for } n\geq 4}{\geq} \frac{n}{2}\log_2 n - \frac{n}{4}\log_2 n = \frac{1}{4} n \log_2 n$$

For all $n \geq 4$. Thus $n \log_2 n \in \mathcal{O}(\log_2(n!))$ is also the case. In general if both $f \in \mathcal{O}(g)$ *and* $g \in \mathcal{O}(f)$ are true, these two functions are called *asymptotically equivalent* in terms of the $\mathcal{O}$-notation. This is denoted by $f \in \Theta(g)$.

(c) Obviously $2^n \leq 3^n$ for all $n \geq 1$. The converse is false though, because a c such that $3^n \leq c2^n$ must fulfill $c \geq (3/2)^n$ for arbitrarily big n, but since $(3/2)^n$ is unbounded there can be no such c.

Exercise 3: Sort Functions by Asymptotic Growth *(6 Points)*

Sort the following functions by asymptotic growth using the $\mathcal{O}$-notation. Write $g <_{\mathcal{O}} f$ if $g \in \mathcal{O}(f)$ *and* $f \notin \mathcal{O}(g)$. Write $g =_{\mathcal{O}} f$ if $f \in \mathcal{O}(g)$ *and* $g \in \mathcal{O}(f)$.

n^2	$\sqrt{n}$	2^n	$\log(n^2)$
3^n	n^{100}	$\log(\sqrt{n})$	$(\log n)^2$
$\log n$	$10^{100}n$	$n!$	$n \log n$
$n \cdot 2^n$	n^n	$\sqrt{\log n}$	n

Sample Solution

	$\sqrt{\log n}$	$<_{\mathcal{O}}$	$\log(\sqrt{n})$	$=_{\mathcal{O}}$	$\log n$	$=_{\mathcal{O}}$	$\log(n^2)$
$<_{\mathcal{O}}$	$(\log n)^2$	$<_{\mathcal{O}}$	$\sqrt{n}$	$<_{\mathcal{O}}$	n	$=_{\mathcal{O}}$	$10^{100}n$
$<_{\mathcal{O}}$	$n \log n$	$<_{\mathcal{O}}$	n^2	$<_{\mathcal{O}}$	n^{100}	$<_{\mathcal{O}}$	2^n
$<_{\mathcal{O}}$	$n \cdot 2^n$	$<_{\mathcal{O}}$	3^n	$<_{\mathcal{O}}$	$n!$	$<_{\mathcal{O}}$	n^n