Exercise 1: Propositional Logic: Basic Terms  \((2+2+2+2 \text{ Points})\)

Let \(\Sigma := \{p, q, r\}\) be a set of atoms. An interpretation \(I : \Sigma \rightarrow \{T, F\}\) maps every atom to either true or false. Inductively, an interpretation \(I\) can be extended to composite formulae \(\varphi\) over \(\Sigma\) (cf. lecture). We write \(I \models \varphi\) if \(\varphi\) evaluates to \(T\) (true) under \(I\). In case \(I \models \varphi\), \(I\) is called a model for \(\varphi\).

For each of the following formulae, give all interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) \(\varphi_1 = (p \land \neg q) \lor (\neg p \lor q)\)

(b) \(\varphi_2 = (\neg p \land (\neg p \lor q)) \leftrightarrow (p \lor \neg q)\)

(c) \(\varphi_3 = (p \land \neg q) \rightarrow \neg(p \land q)\)

(d) \(\varphi_4 = (p \land q) \rightarrow (p \lor r)\)

Remark: \(a \rightarrow b :\equiv \neg a \lor b, \ a \leftrightarrow b :\equiv (a \rightarrow b) \land (b \rightarrow a), \ a \not\rightarrow b :\equiv \neg(a \rightarrow b).\)

**Sample Solution**

(a) See Table 1. The result shows that \(\varphi_1\) is a tautology.

(b) See Table 2. The result shows that \(\varphi_2\) is satisfiable.

(c) \(\varphi_3\) is equivalent to \(\neg(p \land \neg q) \lor (\neg p \lor \neg q)\) which is equivalent to \((\neg p \lor q) \lor (\neg p \lor \neg q)\) which is equivalent to \(\neg p \lor q \lor \neg p \lor \neg q\) which is equivalent to \(\neg p \lor \neg q \lor q\) which is a tautology as either \(q\) or \(\neg q\) holds.

(d) See Table 3. The result shows that \(\varphi_4\) is a tautology.
<table>
<thead>
<tr>
<th>model</th>
<th>p</th>
<th>q</th>
<th>( p \land \neg q )</th>
<th>( \neg p \lor q )</th>
<th>( \varphi_1 )</th>
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Table 1: Truth tables for Exercises 1 (a).

<table>
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<tr>
<th>model</th>
<th>p</th>
<th>q</th>
<th>( \neg p \lor q )</th>
<th>( \neg p \land (\neg p \lor q) )</th>
<th>( p \lor \neg q )</th>
<th>( \varphi_2 )</th>
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</thead>
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Table 2: Truth tables for Exercises 1 (b).

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<th>r</th>
<th>( p \land q )</th>
<th>( p \lor r )</th>
<th>( \varphi_4 )</th>
</tr>
</thead>
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Table 3: Truth tables for Exercises 1 (d).
Exercise 2: CNF and DNF (2+2 Points)

(a) Convert
\[ \psi_1 := (x \land y \rightarrow z \lor w) \land (y \rightarrow x) \land (z \land y \rightarrow 0) \land (w \land y \rightarrow 0) \land y \]
into Conjunctive Normal Form (CNF).

(b) Convert
\[ \psi_2 := \neg((\neg p \leftrightarrow \neg q) \land (\neg r \rightarrow q)) \]
into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step. Note that 2a is not ambiguous as there are clear rules for interpreting such a formula without additional parentheses.

Sample Solution

(a) 
\[ \neg x \lor \neg y \lor z \lor w \land \neg y \lor x \land \neg z \lor \neg y \land \neg w \lor \neg y \land y . \]

(b) 
\[ \neg p \land q \lor \neg q \land p \lor \neg r \land \neg q . \]

Exercise 3: Logical Entailment (2+2 Points)

A knowledge base \( KB \) is a set of formulae over a given set of atoms \( \Sigma \). An interpretation \( I \) of \( \Sigma \) is called a model of \( KB \), if it is a model for all formulae in \( KB \). A knowledge base \( KB \) entails a formula \( \varphi \) (we write \( KB \models \varphi \)), if all models of \( KB \) are also models of \( \varphi \).

Let \( KB := \{ p \lor (q \land \neg r), \neg r \land p \} \). Show or disprove that \( KB \) logically entails the following formulae.

(a) \( \varphi_1 := (p \land q) \lor \neg(\neg r \lor p) \)

(b) \( \varphi_2 := (q \leftrightarrow r) \rightarrow p \)

Sample Solution

(a) \( KB \) does not entail \( \varphi_1 \). Consider the interpretation \( I : p \mapsto 1, q \mapsto 0, r \mapsto 0 \). Interpretation \( I \) is a model for \( KB \) but not for \( \varphi_1 \).

(b) Table 4 shows that every model of \( KB \) is also a model of \( \varphi_2 \), hence \( KB \models \varphi_2 \).

Exercise 4: Inference Rules and Calculi (2+2 Points)

Let \( \varphi_1, \ldots, \varphi_n, \psi \) be propositional formulae. An inference rule
\[ \frac{\varphi_1, \ldots, \varphi_n}{\psi} \]
means that if \( \varphi_1, \ldots, \varphi_n \) are `considered true`, then \( \psi \) is `considered true` as well (\( n = 0 \) is the special case of an axiom). A (propositional) calculus \( C \) is described by a set of inference rules.

Given a formula \( \psi \) and knowledge base \( KB := \{ \varphi_1, \ldots, \varphi_n \} \) (where \( \varphi_1, \ldots, \varphi_n \) are formulae) we write \( KB \models_C \psi \) if \( \psi \) can be derived from \( KB \) by starting from a subset of \( KB \) and repeatedly applying inference rules from the calculus \( C \) to `generate` new formulae until \( \psi \) is obtained.
<table>
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<tr>
<th></th>
<th>model of $KB$</th>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$p \lor (q \land \neg r)$</th>
<th>$\neg r \land p$</th>
<th>$q \leftrightarrow r$</th>
<th>$\varphi_2$</th>
<th>model of $\varphi_2$</th>
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Table 4: Truthtable for Exercise 3 (b).

Consider the following two calculi, defined by their inference rules ($\varphi, \psi, \chi$ are arbitrary formulae).

$$
\text{C}_1 : \quad \frac{\varphi \rightarrow \psi, \psi \rightarrow \chi}{\varphi \rightarrow \psi} \quad \frac{\neg \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \quad \frac{\varphi \leftrightarrow \psi}{\varphi \rightarrow \psi, \psi \rightarrow \varphi}
$$

$$
\text{C}_2 : \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \frac{\varphi \land \psi}{\varphi, \psi} \quad \frac{\neg \varphi \rightarrow \varphi \rightarrow \psi}{(\varphi \land \psi) \rightarrow \chi}
$$

Using the respective calculus, show the following derivations (document your steps).

(a) \{p \leftrightarrow \neg r, \neg q \rightarrow r\} \vdash_{\text{C}_1} p \rightarrow q

(b) \{p \land q, p \rightarrow r, (q \land r) \rightarrow s\} \vdash_{\text{C}_2} s

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Sample Solution

(a) We use $\text{C}_1$ to derive new formulae until we obtain the desired one.

$$
\begin{array}{c}
\neg q \rightarrow r \\
\text{2nd rule} \\
\vdash_{\text{C}_1} \\
\neg r \rightarrow q \\
\text{3rd rule} \\
p \leftrightarrow \neg r \\
\vdash_{\text{C}_1} \\
p \rightarrow \neg r, \neg r \rightarrow p \\
\text{1st rule} \\
p \rightarrow \neg r, \neg r \rightarrow q \\
\vdash_{\text{C}_1} \\
p \rightarrow q
\end{array}
$$

(b) We use $\text{C}_2$ to derive new formulae until we obtain the desired one.

$$
\begin{array}{c}
p \land q \\
\text{2nd rule} \\
\vdash_{\text{C}_2} \\
p, q \\
\text{1st rule} \\
p, p \rightarrow r \\
\vdash_{\text{C}_2} \\
r \\
\text{3rd rule} \\
(q \land r) \rightarrow s \\
\vdash_{\text{C}_2} \\
q \land r \rightarrow s \\
\text{1st rule} \\
r \land r \rightarrow s \\
\vdash_{\text{C}_2} \\
s
\end{array}
$$