University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
P. Schneider

Algorithms and Data Structures
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Exercise Sheet 3

Remark: For this exercise sheet, watch the fourth and fifth video lecture given on the lecture website.

Exercise 1: Case Based Runtime Analysis

The following algorithm obtains a number $x \in \{0, ..., n\}$. Additionally it obtains an array A of size $n+1$ that contains integers $\{0, ..., n\} \setminus \{x\}$ sorted in *ascending* order, whereas the last entry of A is empty. The algorithm inserts x into its position in A and moves the subsequent elements by one position.

Algorithm 1 INSERT($A[0..n]$, x)

```
i ← n
while i ≥ 0 and A[i−1] > x do
    Swap A[i − 1] and A[i]
    i ← i − 1
A[i] ← x
```

Give the runtime of the algorithm *as absolute value and asymptotically* ($\mathcal{O}$-Notation) in the *best, worst and the average case* (compute the average runtime over all possible inputs x).

Remark: To simplify things, assume that one loop cycle takes one time unit, while all other operations have negligible runtime.

Exercise 2: Unsuitable Hash Functions

Let m be the size of a hashtable and let $n \gg m$ be the biggest possible key of any (key,value)-pair. A hash function $h : \{0, \ldots, n\} \to \{0, \ldots, m-1\}$ maps keys to table entries and should meet some criteria to be considered suitable.
The hash function should utilize the whole table, i.e., it should be a surjective function. Furthermore, it should be "chaotic", meaning that it should map similar keys to distinct table entries in order to avoid having lots of collisions in case many similar keys are inserted. A hash function must be deterministic.
The following "hash functions" are unsuitable for various reasons. For each hash function quickly explain why this is the case.

(a) $h_1 : k \mapsto k$. [1]

(b) $h_2 : k \mapsto \lfloor \frac{k}{n} \cdot (m-1) \rfloor$.

(c) $h_3 : k \mapsto 2 \cdot \big(k \bmod \lfloor \frac{m}{2} \rfloor\big)$.

(d) $h_4 : k \mapsto \texttt{random}(m)$, $\big(\texttt{random}(m)$ is picked uniformly at random from $\{0, \ldots, m-1\}\big)$.

[1] The notation $h : k \mapsto h(k)$ means h maps the value k to the value $h(k)$.

Exercise 3: Not a Universal Family of Hash Functions

Let p be prime and $K := \{0, \dots, p-1\}$ be a set of keys. Consider the following set of hash functions.

$$\begin{aligned} \mathbb{H} &:= \left\{ h_{a,b} \mid a, b \in \{0, \dots, p-1\} \right\} \text{ with} \\ h_{a,b}(x) &:= (ax + b) \bmod m, \end{aligned}$$

whereas $p \gg m$.

(a) Show that there exists a subset $S \subseteq K$ of size $\lceil \frac{p}{m} \rceil$ such that $h(x) = h(y)$ for any pair of keys $x, y \in S$ and any function $h \in H$.

(b) Argue that $\mathbb{H}$ is not c-universal for any constant $0 < c < m$.