Algorithms and Data Structures

Levenshtein distance, Dynamic programming

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Algorithms and Data Structures, January 2019

Structure

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Introduction

Edit distance

Edit distance:

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- Measurement for similarity of two words / strings

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- Algorithm for efficient calculation
- General principle: dynamic programming

BioInfSearch

ejafjatlajökuk

eyjafjallajökull

eyjafjallajökull movie

eyjafjallajälull trailer

Search!

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Wikipedia.org:

"Der Eyjafjallajökull (['eɪja,fjatla,jœ:kʏtl̥])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárþing eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."

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Hein Blöd    27568 Bremerhaven
Hein Bloed   27568 Bremerhafen
Hein Doof    27478 Cuxhaven
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- Bioinformatics: Similarity of DNA-sequences

Search of similar proteins:

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- Changed life-science completely

Search of similar proteins:

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- Changed life-science completely
- Cited 63437 times on Google Scholar (Sep. 2017)

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Definition of edit distance: (*Levenshtein-distance*)

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- Let x, y be two strings
- Edit distance $\mathrm{ED}(x,y)$ of x and y:
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- Let x, y be two strings
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Edit distance

Definition of edit distance: (*Levenshtein-distance*)

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  The minimal number of operations to transform x into y
  - `Insert` a character
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  - `Delete` a character

Edit distance
Example

1 2 3 4 5
DOOF

B L OED

1 2 3 4 5
DOOF

$\downarrow$ `replace(1, B)`

BOOF

BLOED

Edit distance

Example

1 2 3 4 5
DOOF

$\downarrow$ `replace(1, B)`

BOOF

$\downarrow$ `replace(2, L)`

B L OF

B L OED

Edit distance
Example

```
1 2 3 4 5
DOOF
  ↓       replace(1, B)
BOOF
  ↓       replace(2, L)
BLOF
  ↓       insert(4, E)
BLOEF

BLOED
```

1 2 3 4 5
DOOF
$\downarrow$ `replace(1, B)`
BOOF
$\downarrow$ `replace(2, L)`
BLOF
$\downarrow$ `insert(4, E)`
BLOEF
$\downarrow$ `replace(5, D)`
BLOED

Edit distance
Example

1 2 3 4 5
DOOF
↓ replace(1, B)
BOOF
↓ replace(2, L)
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↓ insert(4, E)
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↓ replace(5, D)
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$\underbrace{\hspace{6em}}_{\text{ED}=4}$

Edit distance
Example

1 2 3 4 5
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1 2 3 4 5
BLOED

Edit distance
Example

1 2 3 4 5
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BLOEF
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BLOED

$\underbrace{\texttt{replace(1, B)}\;\texttt{replace(2, L)}\;\texttt{insert(4, E)}\;\texttt{replace(5, D)}}_{\text{ED}=4}$

1 2 3 4 5
B LOED

DOOF

1 2 3 4 5			
DOOF			
↓	`replace(1, B)`	1 2 3 4 5	
BOOF		B LOED	
↓	`replace(2, L)`	↓	`replace(5, F)`
BLOF		B LOEF	
↓	`insert(4, E)`		
BLOEF			
↓	`replace(5, D)`	DOOF	
BLOED			

$\underbrace{\qquad\qquad\qquad}_{\text{ED=4}}$

Edit distance

Example

12345
DOOF
↓ `replace(1, B)`
BOOF
↓ `replace(2, L)`
BLOF
↓ `insert(4, E)`
BLOEF
↓ `replace(5, D)`
BLOED

$\underbrace{\hspace{6em}}_{\text{ED=4}}$

12345
B LOED
↓ `replace(5, F)`
B LOEF
↓ `delete(4)`
B LOF

DOOF

1 2 3 4 5 DOOF	
↓	`replace(1, B)`
BOOF	
↓	`replace(2, L)`
BLOF	
↓	`insert(4, E)`
BLOEF	
↓	`replace(5, D)`
BLOED	

$\underbrace{\hspace{4cm}}_{\text{ED=4}}$

1 2 3 4 5 BLOED	
↓	`replace(5, F)`
BLOEF	
↓	`delete(4)`
BLOF	
↓	`replace(2, O)`
BOOF	
DOOF	

Edit distance
Example

1 2 3 4 5	
DOOF	
↓	`replace(1, B)`
BOOF	
↓	`replace(2, L)`
BLOF	
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BLOED	

$\underbrace{\hspace{4cm}}_{\text{ED=4}}$

1 2 3 4 5	
BLOED	
↓	`replace(5, F)`
BLOEF	
↓	`delete(4)`
BLOF	
↓	`replace(2, O)`
BOOF	
↓	`replace(1, D)`
DOOF	

Edit distance
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BLOED	
	ED=4

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BOOF	
$\downarrow$	`replace(1, D)`
DOOF	
	ED=4

Edit distance

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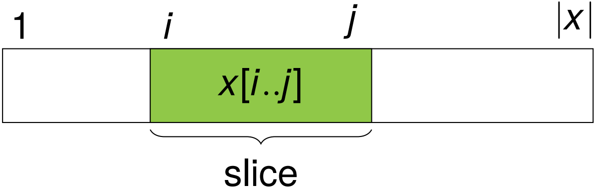

Trivial facts:

Edit distance

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- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$
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- $\mathrm{ED}(x,y) \geq \mathrm{abs}(|x| - |y|)$

$$\mathrm{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

Edit distance

Trivial facts:

- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$
- $\mathrm{ED}(x,\varepsilon) = |x|$
- $\mathrm{ED}(x,y) \geq \mathrm{abs}(|x| - |y|)$ $\quad \mathrm{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$
- $\mathrm{ED}(x,y) \leq \mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ $\quad n = |x|,\ m = |y|$

Solutions based on examples:

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- From `VERIEN` to `FERIEN`?

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- From `MEXIKO` to `AMERIKA`?

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- From `AAEBEAABEAREEAEBA` to `RBEAAEEBAAAEBBAEAE`?

Solutions based on examples:

- From VERIEN to FERIEN?
- From MEXIKO to AMERIKA?
- From AAEBEAABEAREEAEBA to RBEAAEEBAAAEBBAEAE?
- Searching biggest substrings can yield the solution but doesn't have to

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Recursive approach:

Edit distance
Solving examples

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Recursive approach:

- Dividing in two halves? Not a good idea:

$\mathrm{ED}(\mathit{GRAU}, \mathit{RAUM}) = 2$ but $\mathrm{ED}(\mathit{GR}, \mathit{RA}) + \mathrm{ED}(\mathit{AU}, \mathit{UM}) = 4$

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Recursive approach:

- Dividing in two halves? Not a good idea:

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- Finding "smaller" sub problems?
  Let's try it!

Terminology:

Edit distance

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- Let x, y be two strings

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Terminology:

- Let x, y be two strings
- Let $\sigma_1, \ldots, \sigma_k$ be a sequence of k operations where $k = \mathrm{ED}(x, y)$ for $x \rightarrow y$ (transform x into y)
  (We do not know this sequence but we assume it exists)

Terminology:

Edit distance

Terminology:

- We only consider monotonous sequences:
  The posititition of σ_{i+1} is $\geq$ the position of σ_i where we only allow the positions to be equal on a `delete` operation

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DOOF		SAUDOOF	
↓	`replace(1, B)`	↓	`delete(1)`
BOOF		AUDOOF	
↓	`replace(2, L)`	↓	`delete(1)`
BLOF		UDOOF	
↓	insert(4, E)	↓	`delete(1)`
BLOEF		DOOF	
↓	`replace(5, D)`	↓	`insert(4, O)`
BLOED		DOOOF	

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  The positition of σ_{i+1} is $\geq$ the position of σ_i where we only allow the positions to be equal on a `delete` operation

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BLOED		DOOOF	

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1	2	3	4	5
D	O	O	F	
B	L	O	E	D

1	2	3	4	5	6	7
S	A	U	D	O	O	F
D	O	O	O	F		

Consider the last operation:

Consider the last operation:

- Solve blue part recursively

Consider the last operation:

- Solve blue part recursively

DOOF
↓↓↓↓
BLOE
↓insert
BLOED

Figure: Case 1a

DOOF
↓↓↓↓↓↓
BLOEDF
↓delete
BLOED

Figure: Case 1b

DOOF
↓↓↓↓↓
BLOEF
↓replace
BLOED

Figure: Case 1c

Consider the last operation:

Consider the last operation:

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Consider the last operation:

- Solve blue part recursively

WINTER

↓↓↓↓↓↓

SOMMER

↓nothing

SOMMER

Figure: Case 2

Display of solution:

- Alignment
- Example:

_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

Dynamic programming:

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- Instances of Bellman's principle of optimality:

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  - Shortest paths

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Figure: Richard Bellman (1920 - 1984)

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Dynamic programming:

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Figure: Richard Bellman (1920 - 1984)

- Optimal solutions consist of optimal partial solutions
  - Shortest paths: Each partial path has to be optimal
  - Edit distance: Each partial alignment has to be optimal

- Always solvable through dynamic programming (Caching of optimal partial solutions)

Case analysis:

Edit distance

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- We consider the last operation σ_k

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Case analysis:

- We consider the last operation σ_k
  - $\sigma_1, \ldots, \sigma_{k-1}$: $x \to z$ and σ_k: $z \to y$
    Example:

$$x = \text{DOOF},\ z = \text{SAUBLOEF},\ y = \text{SAUBLOED}$$

Edit distance

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    Example:

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  - Let $n = |x|$, $m = |y|$, $m' = |z|$

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    Example:

$$x = \mathrm{DOOF},\ z = \mathrm{SAUBLOEF},\ y = \mathrm{SAUBLOED}$$

  - Let $n = |x|$, $m = |y|$, $m' = |z|$
  - We note $m' \in \{m-1, m, m+1\}$ why?

Case analysis:

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  - Case 1a: $\sigma_k = \mathtt{insert}(m' + 1, y[m])$ [then $m' = m - 1$]
  - Case 1b: $\sigma_k = \mathtt{delete}(m')$ [then $m' = m + 1$]

Edit distance

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- Case 1: σ_k does something at the outer end:
  - Case 1a: $\sigma_k = \texttt{insert}(m'+1, y[m])$ [then $m' = m-1$]
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  - Case 1c: $\sigma_k = \texttt{replace}(m', y[m])$ [then $m' = m$]
- Case 2: σ_k does nothing at the outer end:
  - Then $z[m'] = y[m]$ and $x[n'] = z[m']$ and with that $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$ and $x[n] = y[m]$

Edit distance

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- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$

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- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$
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- Case 1c (`replace`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1, \ldots, \sigma_k$: $x[1..n-1] \rightarrow y[1..m-1]$

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This results in the recursive formula:

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This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $\mathrm{ED}(x,y)$ the minimum of
  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y) + 1$ and

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y$
- Case 1c (`replace`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1, \ldots, \sigma_k$: $x[1..n-1] \rightarrow y[1..m-1]$

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  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$

Edit distance

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- Case 1b (`delete`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y$
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  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$

Edit distance

Case analysis:

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  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $\mathrm{ED}(x,y) = |y|$

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Case analysis:

- Case 1a (`insert`): $\sigma_1,\ldots,\sigma_{k-1}\colon x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1,\ldots,\sigma_{k-1}\colon x[1..n-1] \rightarrow y$
- Case 1c (`replace`): $\sigma_1,\ldots,\sigma_{k-1}\colon x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1,\ldots,\sigma_k\colon x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $\mathrm{ED}(x,y)$ the minimum of
  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $\mathrm{ED}(x,y) = |y|$
- For $|y| = 0$ is $\mathrm{ED}(x,y) = |x|$

Edit distance
Implementation - Python

```
def edit_distance(x, y):
    if len(x) == 0:
        return len(y)
    if len(y) == 0:
        return len(x)

    ed1 = edit_distance(x, y[:-1]) + 1
    ed2 = edit_distance(x[:-1], y) + 1
    ed3 = edit_distance(x[:-1], y[:-1])
    if x[-1] != y[-1]:
        ed3 += 1

    return min(ed1, ed2, ed3)
```

Recursive program:

Recursive program:

- The algorithm results in the following recursive formular:

$$\begin{aligned} T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\ &\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\ &= 3 \cdot T(n-1,m-1) \end{aligned}$$

Recursive program:

- The algorithm results in the following recursive formular:

$$
\begin{aligned}
T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\
&\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\
&= 3 \cdot T(n-1,m-1)
\end{aligned}
$$

- This results in $T(n,n) \geq 3^n$

Recursive program:

- The algorithm results in the following recursive formular:

$$\begin{aligned} T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\ &\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\ &= 3 \cdot T(n-1,m-1) \end{aligned}$$

- This results in $T(n,n) \geq 3^n$

$\Rightarrow$ The runtime is at least exponential

Dynamic programming:

Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

Edit distance

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- We create a table with all possible combination of substrings and save calculated entries
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Visualization on the next slide:

Edit distance

Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

Visualization on the next slide:

- Operations always refer to the last position (indices are omitted)

Edit distance

Dynamic programming:

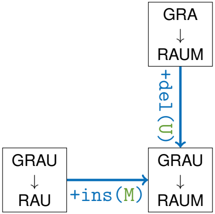
- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

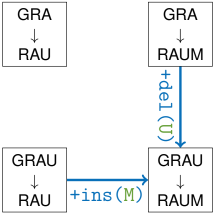
Visualization on the next slide:

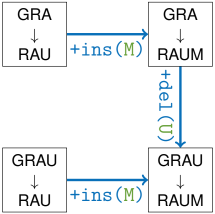
- Operations always refer to the last position (indices are omitted)
- We also display the replaced character on a `replace` operation to visualize operations without costs $\Rightarrow$ `repl(A, A)`

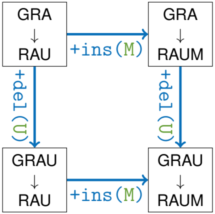
GRAU
$\downarrow$
RAUM

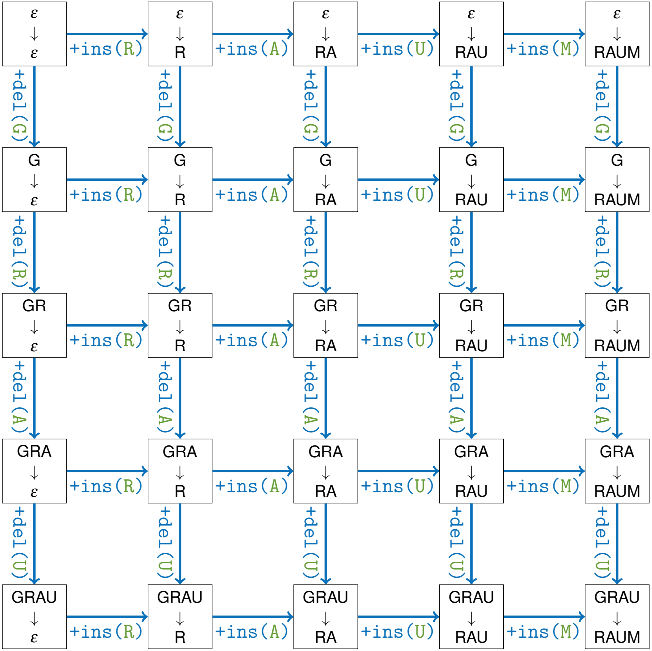
$$\boxed{\begin{array}{c}\text{GRAU}\\ \downarrow \\ \text{RAU}\end{array}} \xrightarrow{\texttt{+ins(M)}} \boxed{\begin{array}{c}\text{GRAU}\\ \downarrow \\ \text{RAUM}\end{array}}$$


GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM


GRA ↓ RAU
GRA ↓ RAUM
+del(U)
GRAU ↓ RAU
+ins(M)
GRAU ↓ RAUM

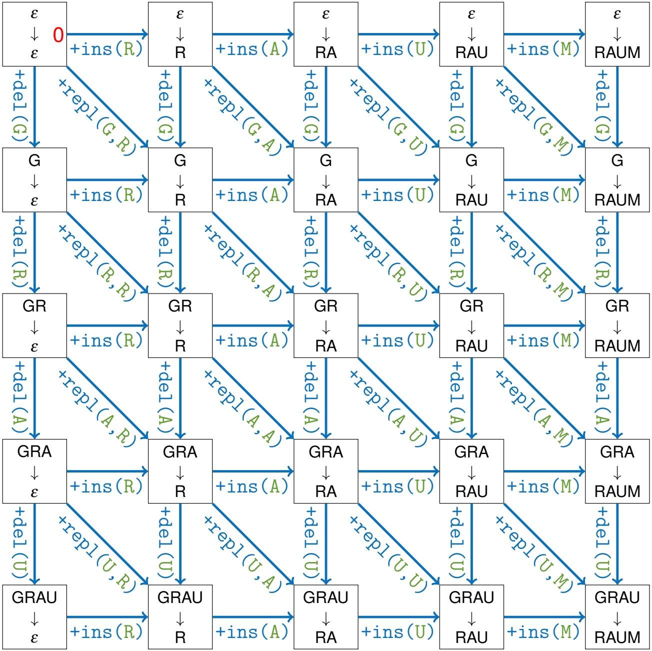

GRA
RAU
+ins(M)
GRA
RAUM
+del(U)
GRAU
RAU
+ins(M)
GRAU
RAUM

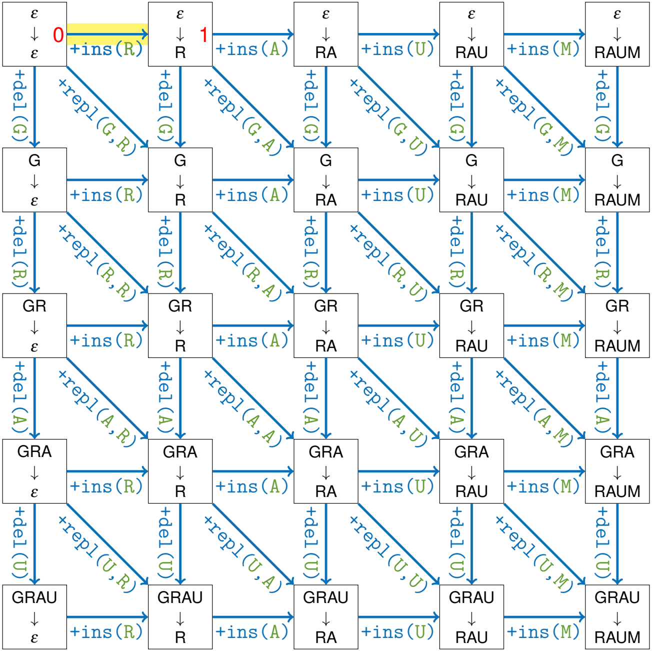

GRA → RAU
+ins(M)
GRA → RAUM
+del(U)
+del(U)
GRAU → RAU
+ins(M)
GRAU → RAUM



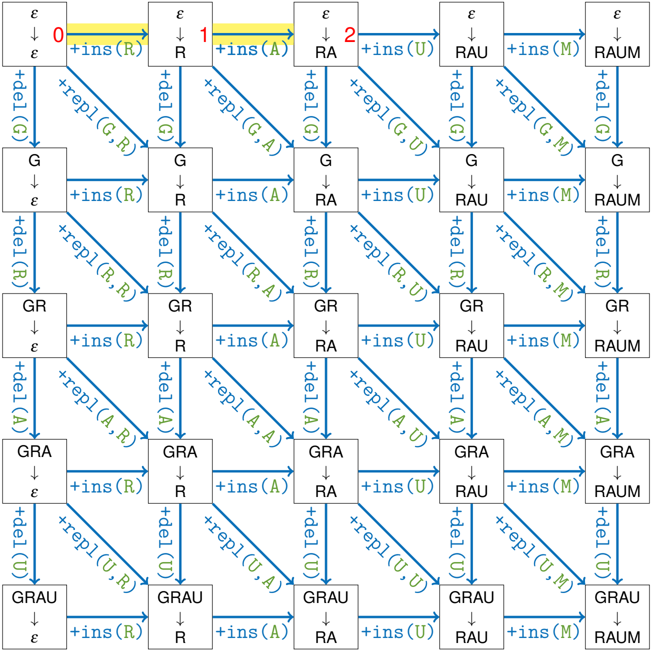
ε ↓ ε
+ins(R)
ε ↓ R
+ins(A)
ε ↓ RA
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
+del(G)
+del(G)
+del(G)
+del(G)
G ↓ ε
+ins(R)
G ↓ R
+ins(A)
G ↓ RA
+ins(U)
G ↓ RAU
+ins(M)
G ↓ RAUM
+del(R)
+del(R)
+del(R)
+del(R)
+del(R)
GR ↓ ε
+ins(R)
GR ↓ R
+ins(A)
GR ↓ RA
+ins(U)
GR ↓ RAU
+ins(M)
GR ↓ RAUM
+del(A)
+del(A)
+del(A)
+del(A)
+del(A)
GRA ↓ ε
+ins(R)
GRA ↓ R
+ins(A)
GRA ↓ RA
+ins(U)
GRA ↓ RAU
+ins(M)
GRA ↓ RAUM
+del(U)
+del(U)
+del(U)
+del(U)
+del(U)
GRAU ↓ ε
+ins(R)
GRAU ↓ R
+ins(A)
GRAU ↓ RA
+ins(U)
GRAU ↓ RAU
+ins(M)
GRAU ↓ RAUM

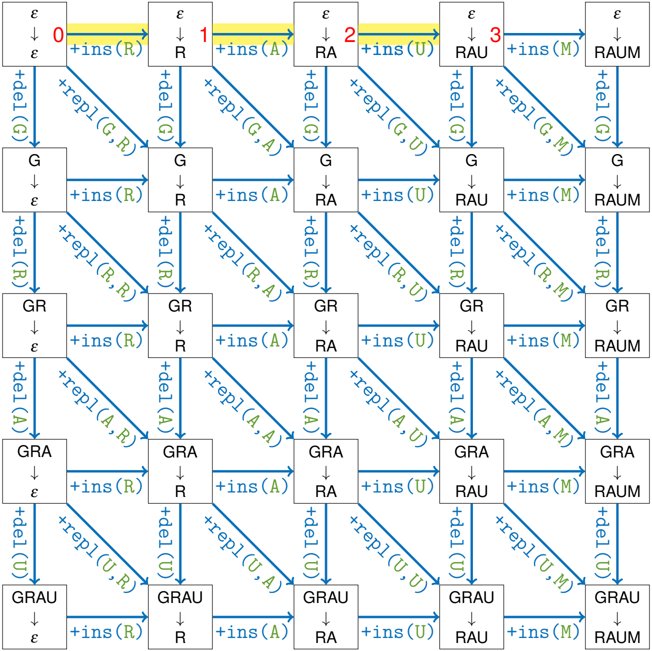
Fast algorithm:
We can determine the edit distance for all combination of partial strings from the top left to bottom right.


ε ↓ ε
0
+ins(R)
+ins(A)
+ins(U)
+ins(M)
ε ↓ R
ε ↓ RA
ε ↓ RAU
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

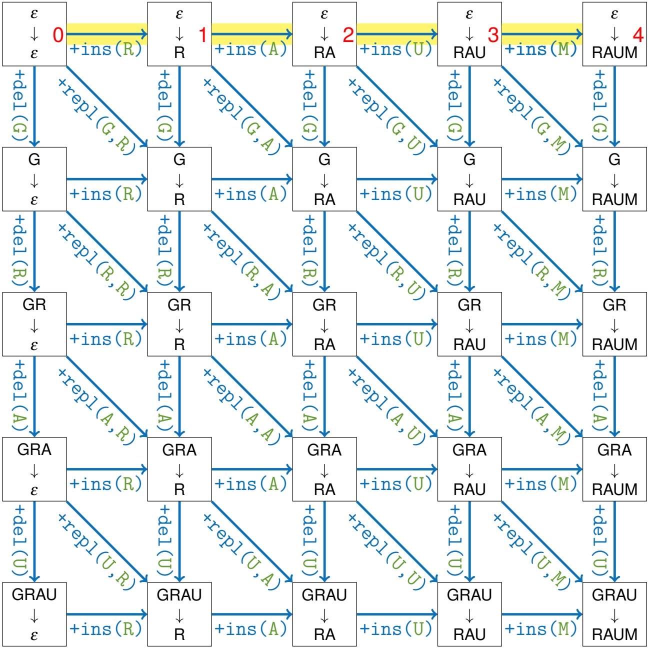


ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

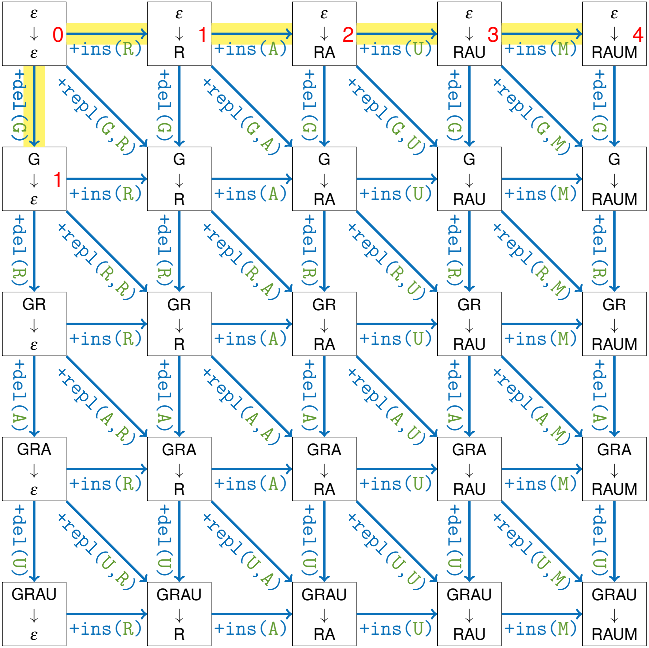

ε → ε
ε → R
ε → RA
ε → RAU
ε → RAUM
0
1
2
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
G → ε
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM

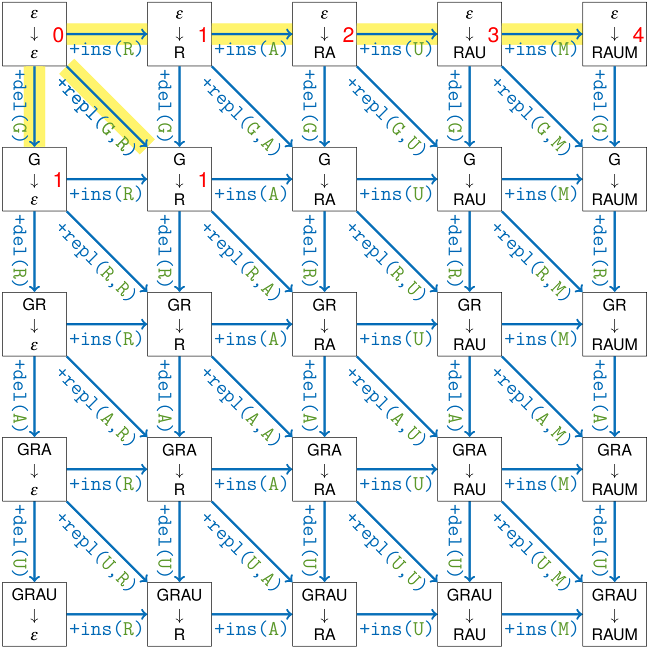


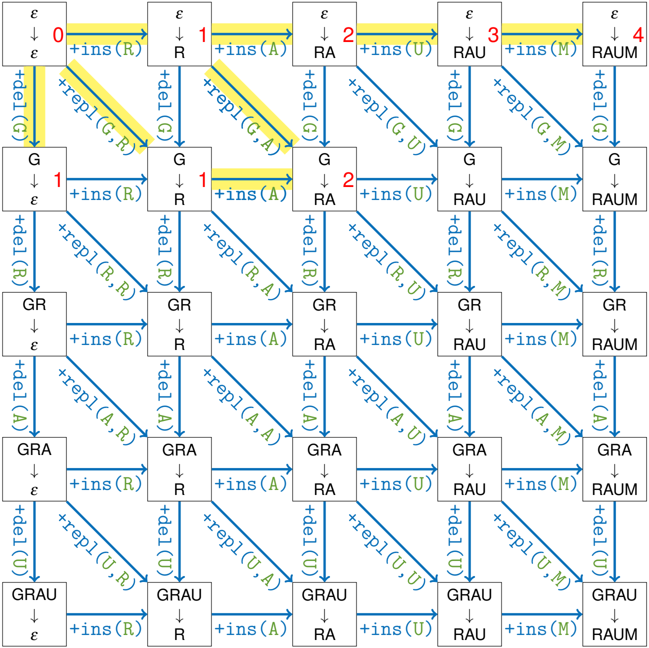
ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
3
+ins(M)
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
+ins(R)
G ↓ R
+ins(A)
G ↓ RA
+ins(U)
G ↓ RAU
+ins(M)
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
+ins(R)
GR ↓ R
+ins(A)
GR ↓ RA
+ins(U)
GR ↓ RAU
+ins(M)
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
+ins(R)
GRA ↓ R
+ins(A)
GRA ↓ RA
+ins(U)
GRA ↓ RAU
+ins(M)
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
+ins(R)
GRAU ↓ R
+ins(A)
GRAU ↓ RA
+ins(U)
GRAU ↓ RAU
+ins(M)
GRAU ↓ RAUM

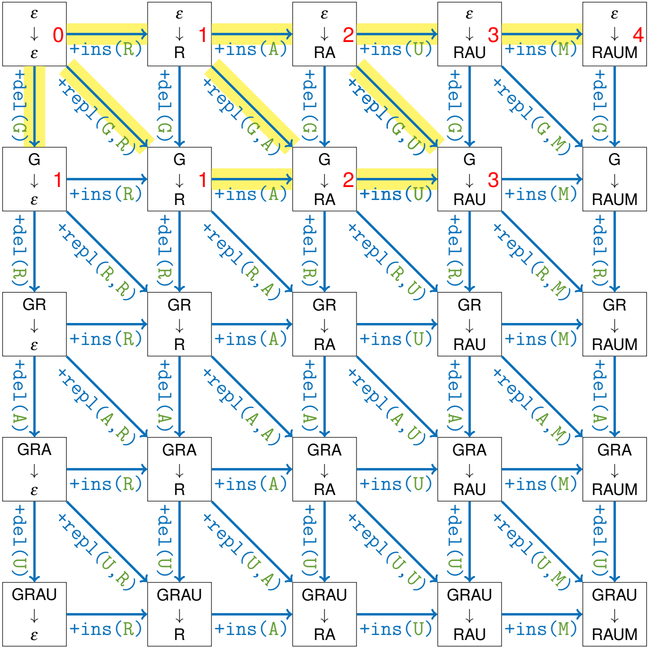


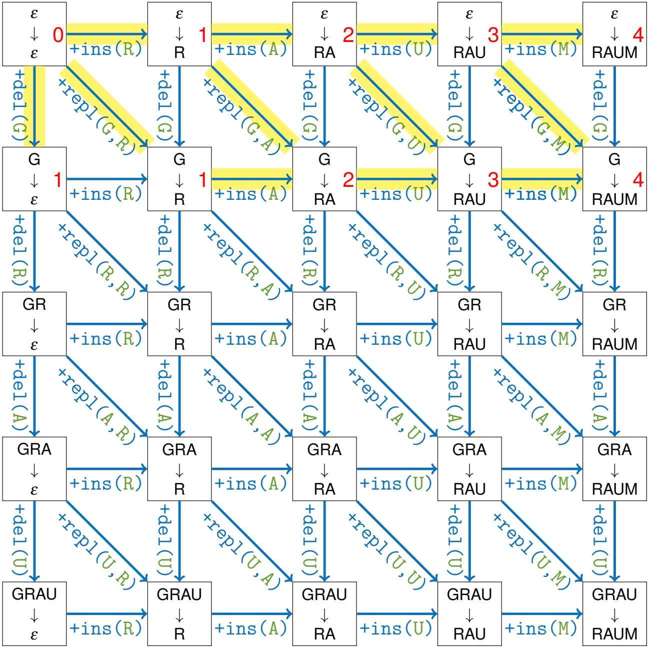
ε → ε
ε → R
ε → RA
ε → RAU
ε → RAUM
0
1
2
3
4
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
G → ε
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM

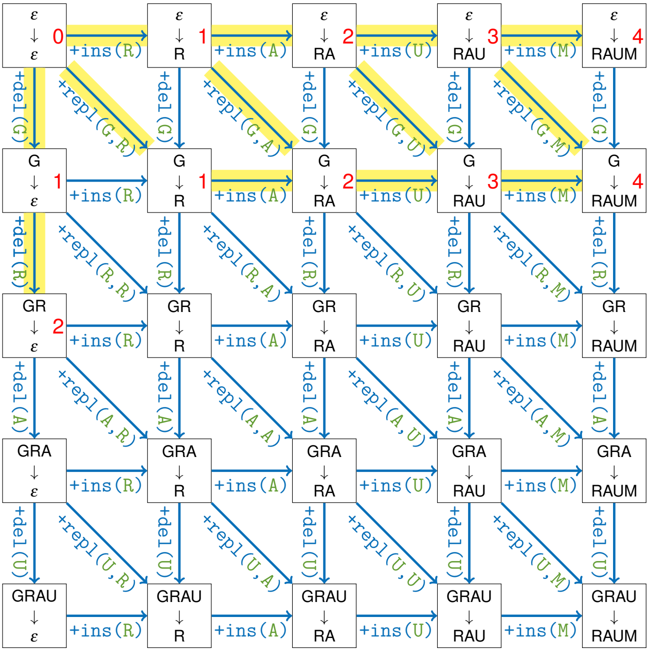

ε → ε
ε → R
ε → RA
ε → RAU
ε → RAUM
G → ε
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
0
1
2
3
4
1
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

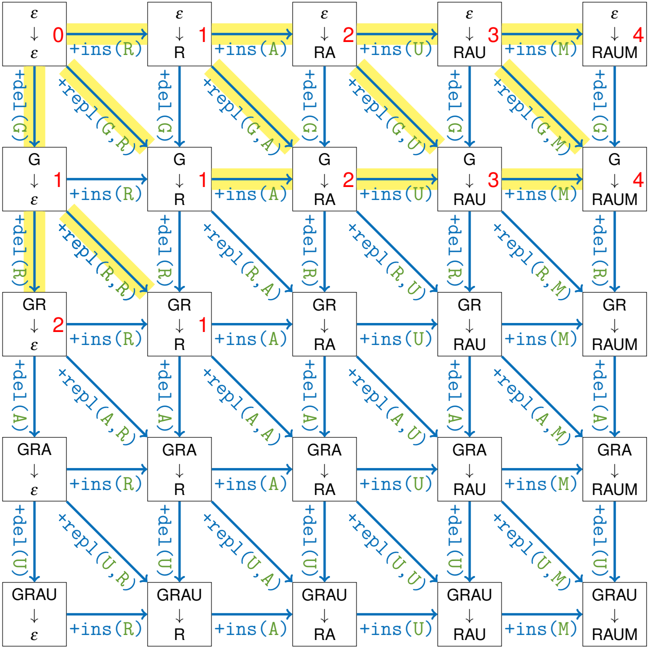

ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
3
+ins(M)
ε ↓ RAUM
4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
1
G ↓ R
1
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε
ε ↓ R
ε ↓ RA
ε ↓ RAU
ε ↓ RAUM
0
1
2
3
4
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
1
1
2
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε → ε 0
+ins(R)
ε → R 1
+ins(A)
ε → RA 2
+ins(U)
ε → RAU 3
+ins(M)
ε → RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM


ε
ε
R
RA
RAU
RAUM
G
GR
GRA
GRAU
0
1
2
3
4
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)



ε → ε
ε → R
ε → RA
ε → RAU
ε → RAUM
G → ε
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
0
1
2
3
4
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
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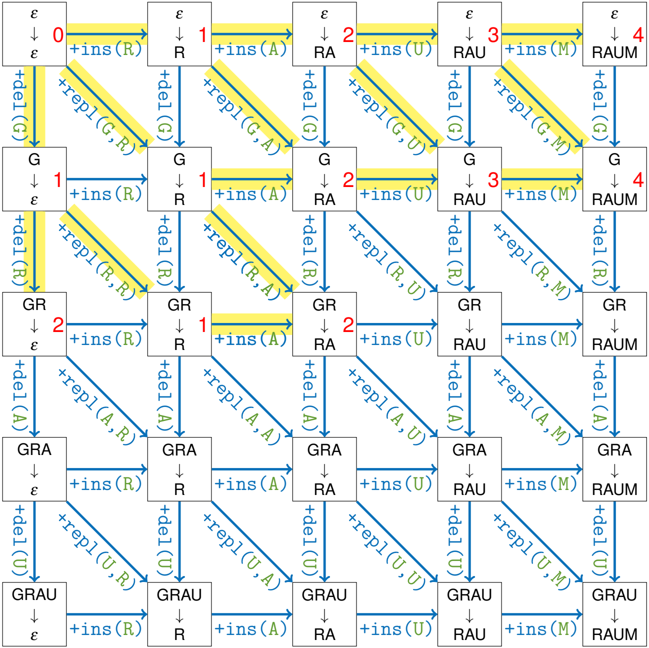
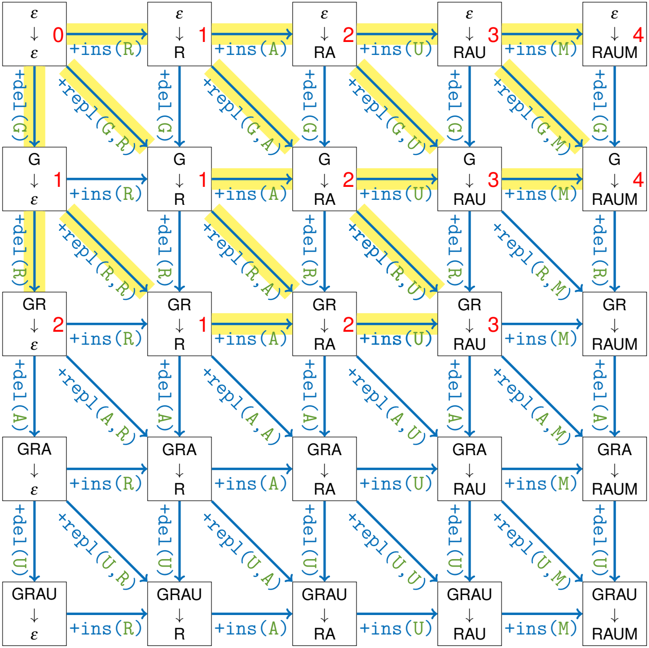
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)



ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
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+ins(U)
+ins(M)
+del(G)
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+del(A)
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+repl(G,R)
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+repl(G,M)
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+repl(R,A)
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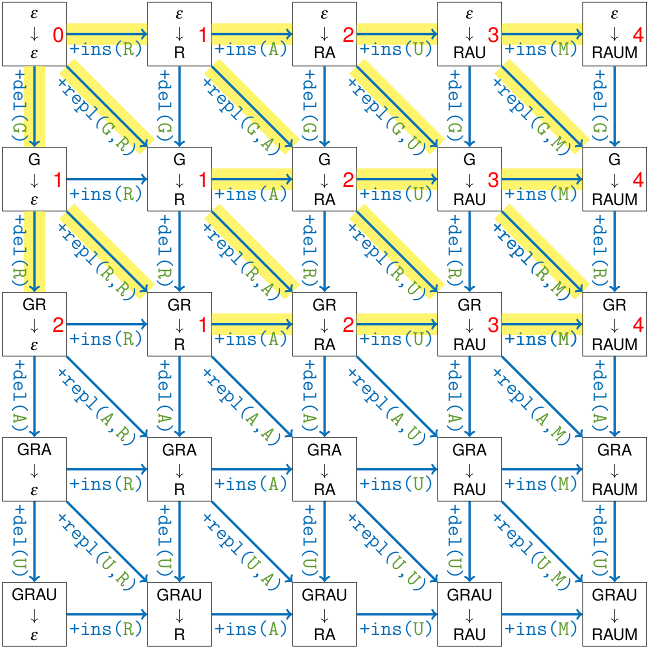
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM 4
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
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+ins(U)
+ins(M)
+del(G)
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+del(U)
+repl(G,R)
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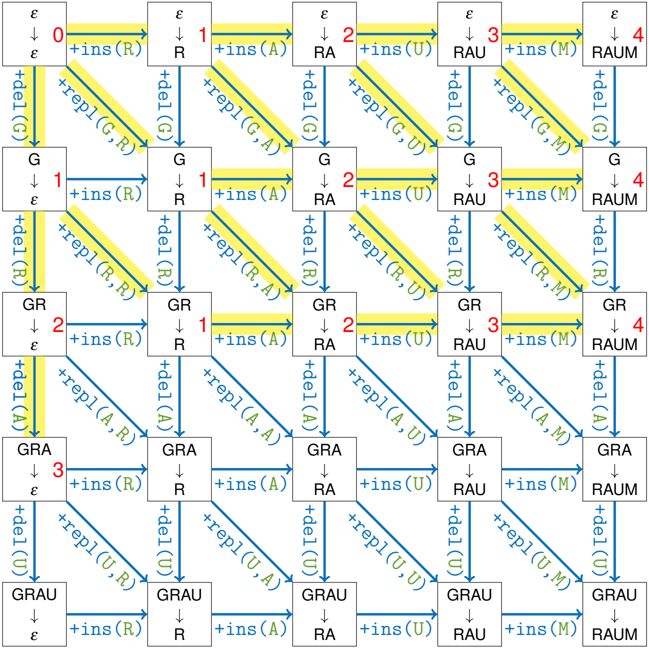
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM 4
GRA → ε 3
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
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+ins(U)
+ins(M)
+del(G)
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+del(U)
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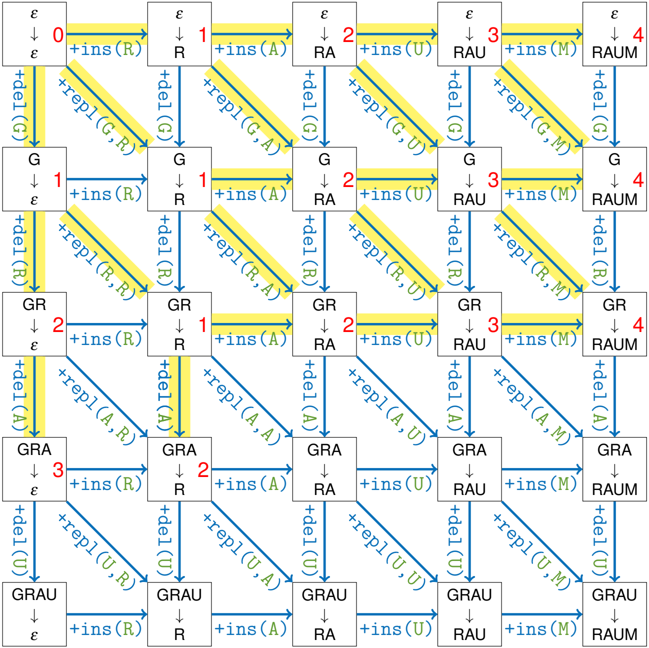
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)



ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
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+repl(G,M)
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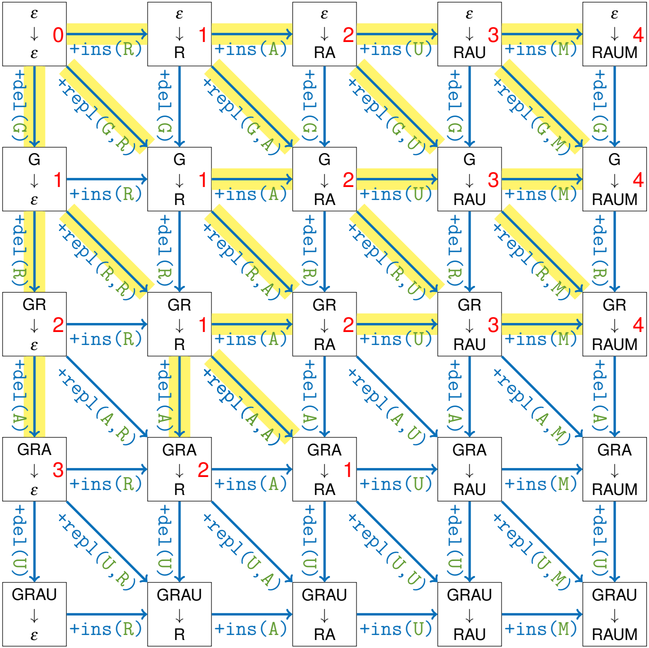
+repl(U,U)
+repl(U,M)


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
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+repl(G,M)
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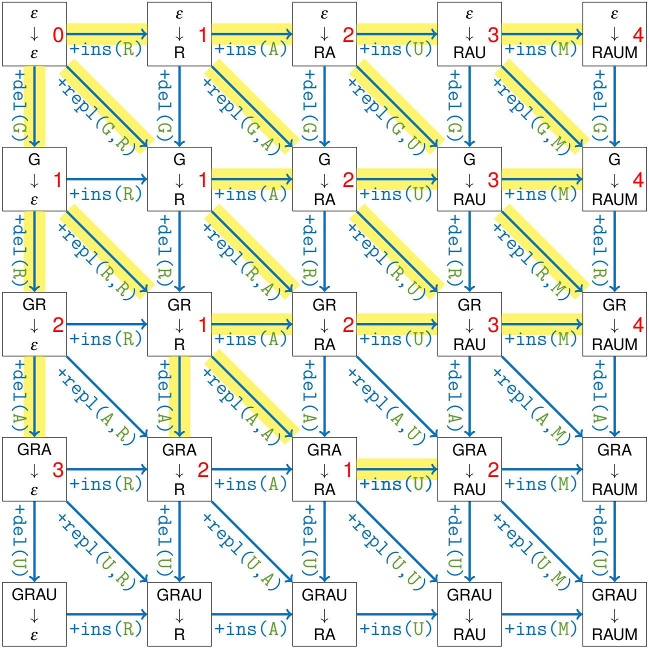
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
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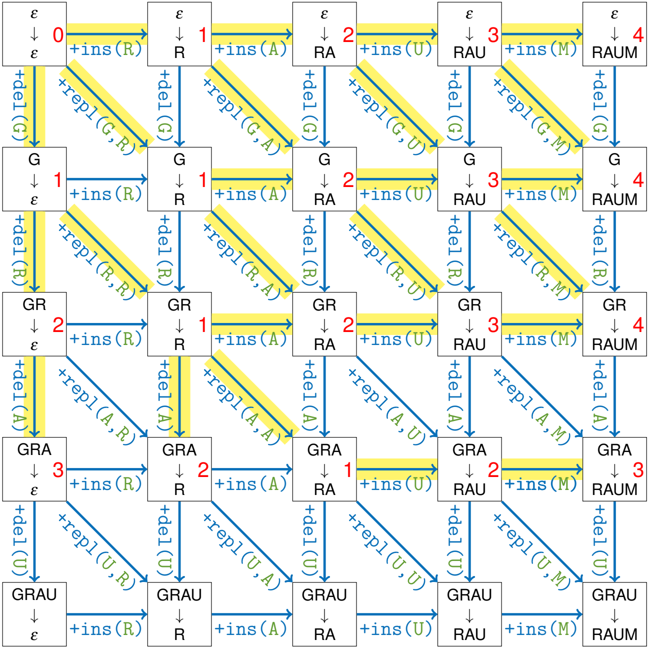
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)



ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM 4
GRA → ε 3
GRA → R 2
GRA → RA 1
GRA → RAU 2
GRA → RAUM 3
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
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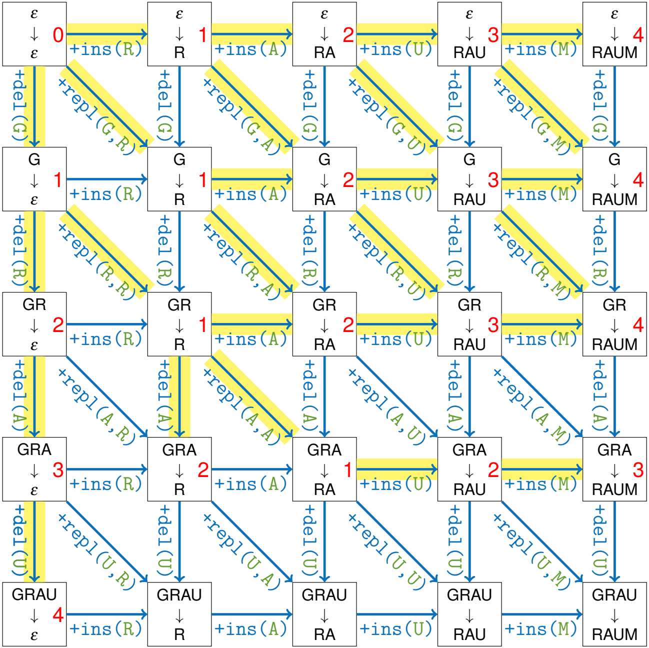
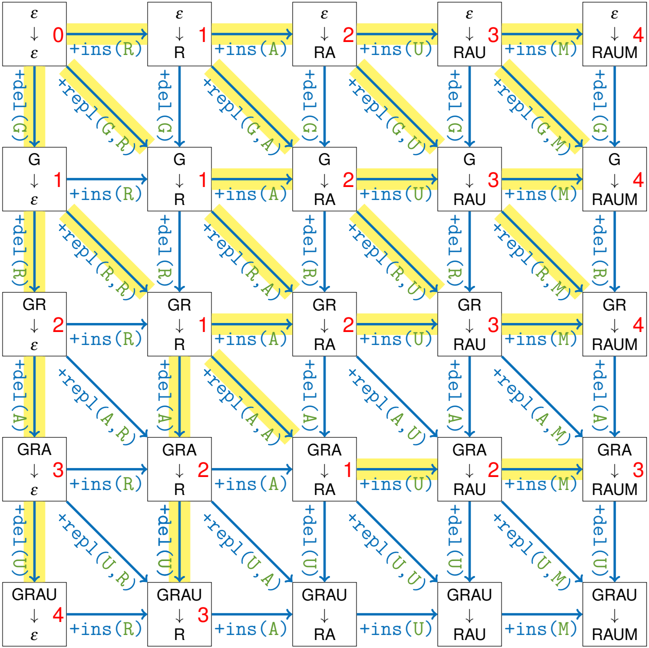
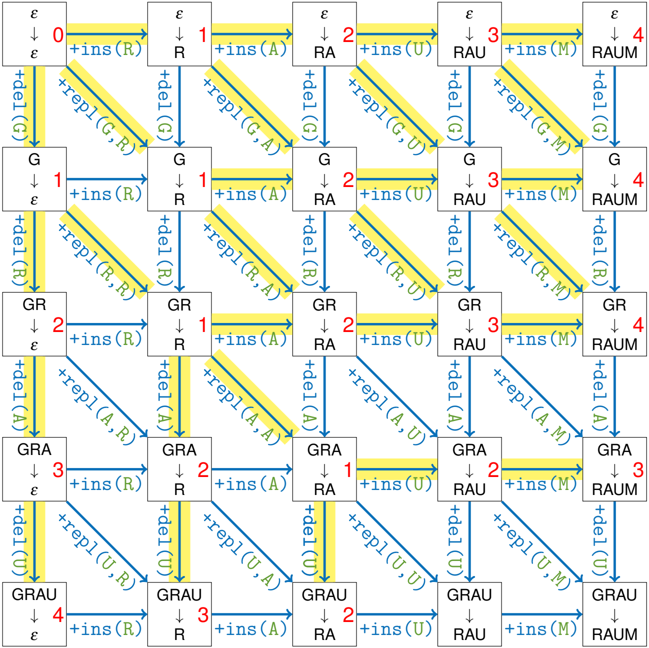
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM 4
GRA → ε 3
GRA → R 2
GRA → RA 1
GRA → RAU 2
GRA → RAUM 3
GRAU → ε 4
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
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+repl(A,A)
+repl(A,U)
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+repl(U,R)
+repl(U,A)
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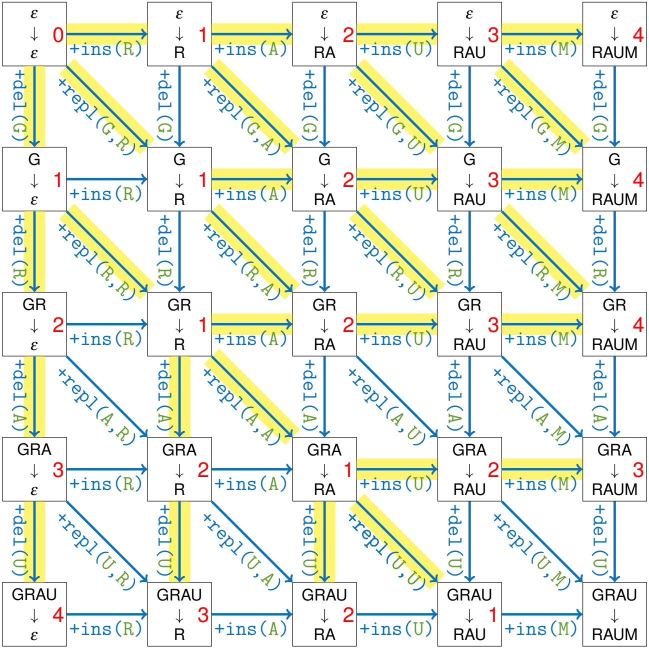
+repl(U,M)



ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
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+del(R)
+del(A)
+del(U)
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+repl(G,M)
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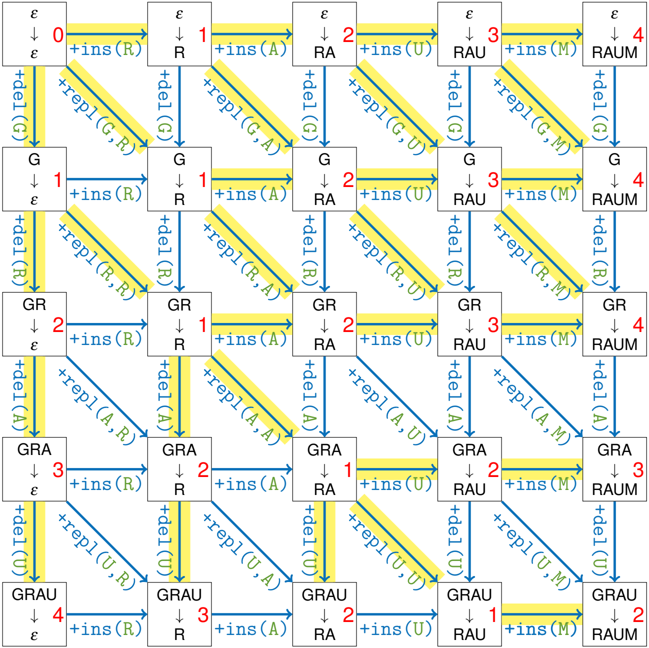

ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM 2
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
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+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

How to get the sequence of operations?

How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)

How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)
- There can be more than one arrows to the three previous entries

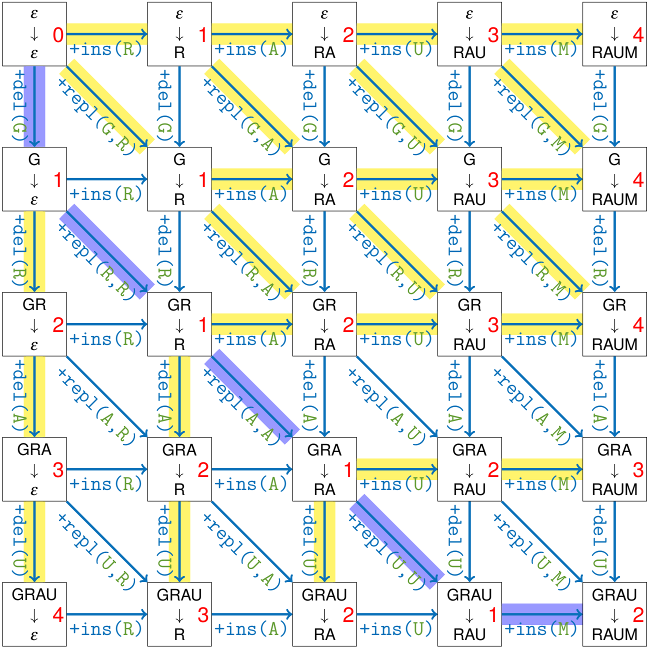
How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)
- There can be more than one arrows to the three previous entries
- If we follow the highlighted path from (n, m) to $(1, 1)$ we get the optimum operations to transform x into y

Edit distance

How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)
- There can be more than one arrows to the three previous entries
- If we follow the highlighted path from (n, m) to $(1, 1)$ we get the optimum operations to transform x into y
  - If we can follow more than one path there exist more than one ideal sequence


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM 2
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

General principle:

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- Recursive computation of …
  - … the same reoccuring partial problems
  - … a limited number of partial problems

Edit distance

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  - … the same reoccuring partial problems
  - … a limited number of partial problems
- Computation of the solutions for all partial problems
- In a order that unsolved partial problems consist of already solved partial problems
- The "path" to our solution normally gets computed while searching the best solution
- Dijkstra algorithm is basically dynamic programming!

Additional applications:

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- *Edit distance*: global alignment with $O(n^2)$ space and time consumption

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_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

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S	A	U	B	L	O	E	D

  - Solution in $O(n^3)$ time or $O(n^2)$ affine

$O(n^2)$ space consumption might be problematic:

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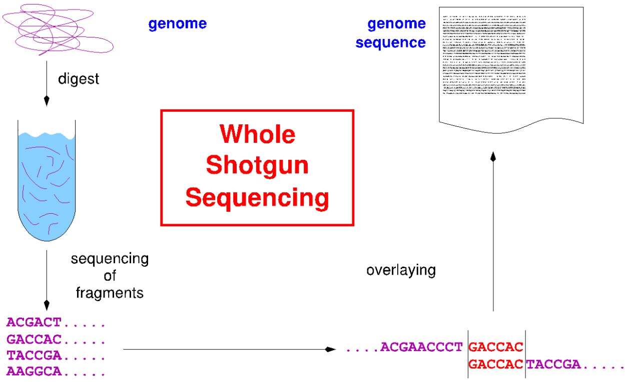
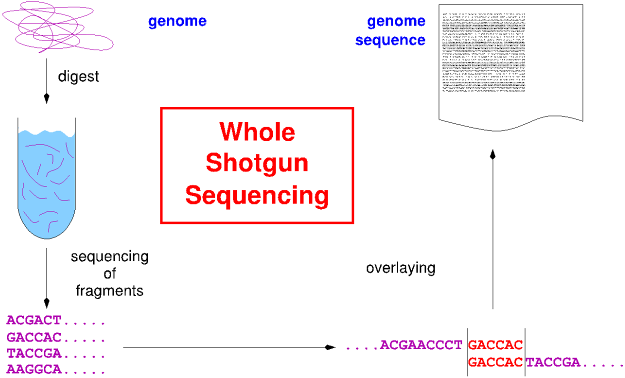
- Divide-and-conquer approach
- $O(n)$ space and $O(n^2)$ time consumption

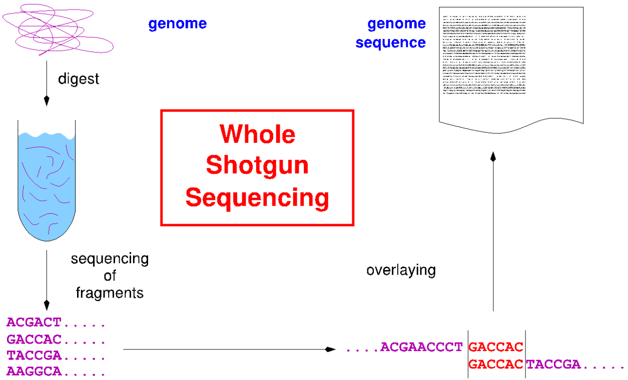
Edit distance

Additional applications (III)





- Sequencing: $O(n^2)$ is too much



- Sequencing: $O(n^2)$ is too much
- Index: suffixtree, suffixarray, burrow-wheeler-transform

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Dynamic programming**

  [Wik] Dynamic programming
  https://en.wikipedia.org/wiki/Dynamic_programming

- **Edit distance**

  [Wik] Levenshtein distance
  https://en.wikipedia.org/wiki/Levenshtein_distance