University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
P. Schneider

Algorithms and Data Structures
Winter Term 2019/2020
Sample Solution Exercise Sheet 2

Remark: For this exercise sheet, watch the third video lecture given on the lecture website.

Exercise 1: $\mathcal{O}$-Notation

State whether the following claims are correct or not and prove it using the formal definition of the $\mathcal{O}, \Omega, \Theta$-notations.

(a) $n(n-1) \in \mathcal{O}(n^2)$

(b) $n! \in \Omega\big(n^2\big)$

(c) $n \in \Theta(\log_2 3^n)$

(d) $\sqrt{n^3} \in \mathcal{O}(n \log n)$ ***Hint:*** *For all $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$: $\log_2 n \leq n^{\varepsilon}$.*

Sample Solution

(a) The claim is true. This is because $n(n-1) \leq n \cdot n = n^2$ for any $n \in \mathbb{N}$.

(b) The claim is true. We choose $n_0 = 4$ and $c = 1$. Then we have $n^2 \leq n \cdot (n-1) \cdot 2 \leq n!$ for all $n \geq 4$.

(c) The statement is true. Choose $c = \log_2 3$ und $n_0 = 1$. Then for all $n \geq n_0$ we have

$$\log_2 3^n = n \cdot \log_2 3 = c \cdot n.$$

The above equation implies both $\log_2 3^n \in \mathcal{O}(n)$ and $\log_2 3^n \in \Omega(n)$, hence $\log_2 3^n \in \Theta(n)$ (and by symmetry also $n \in \Theta(\log_2 3^n)$).

(d) The claim is false. Assume there exist $c > 0, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$: $\sqrt{n^3} \leq cn \log n$.

$$\begin{aligned}
& \sqrt{n^3} \leq cn \log n, \quad \forall n \geq n_0 \\
\iff & n^{1/2} \leq c \log n, \quad \forall n \geq n_0 \\
\iff & n^{1/4} \cdot n^{1/4} \leq c \log n, \quad \forall n \geq n_0 \\
\implies & \big(n^{1/4} \leq c \quad \text{OR} \quad n^{1/4} \leq \log n\big), \quad \forall n \geq n_0
\end{aligned}$$

But $n^{1/4} \leq c$ is contradictory for all $n \geq c^4$. Additionally $n^{1/4} \leq \log n \ \forall n \geq n_0$ is a contradiction to the hint. Thus the assumption must have been false, and therefore $\sqrt{n^3} \notin \mathcal{O}(n \log n)$.

Exercise 2: Sort Functions by Asymptotic Growth

Use the definition of the $\mathcal{O}$-notation to give a sequence of the functions below, which is ordered by asymptotic growth (ascending). Between two consecutive functions g and f in your sequence, insert either $\prec$ (in case $g \in \mathcal{O}(f)$ *and* $f \notin \mathcal{O}(g)$) or $\simeq$ (in case $g \in \mathcal{O}(f)$ *and* $f \in \mathcal{O}(g)$).

n^2	$\sqrt{n}$	$2^{\sqrt{n}}$	$\log(n^2)$
$2^{\sqrt{\log_2 n}}$	$\log(n!)$	$\log(\sqrt{n})$	$(\log n)^2$
$\log n$	$10^{100}n$	$n!$	$n \log n$
$2^n/n$	n^n	$\sqrt{\log n}$	n

Sample Solution

	$\sqrt{\log n}$	$\prec$	$\log(\sqrt{n})$	$\simeq$	$\log n$	$\simeq$	$\log(n^2)$
$\prec$	$(\log n)^2$	$\prec$	$2^{\sqrt{\log_2 n}}$	$\prec$	$\sqrt{n}$	$\prec$	n
$\simeq$	$10^{100}n$	$\prec$	$n \log n$	$\simeq$	$\log(n!)$	$\prec$	n^2
$\prec$	$2^{\sqrt{n}}$	$\prec$	$2^n/n$	$\prec$	$n!$	$\prec$	n^n