Chapter 1
Divide and Conquer

Algorithm Theory
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Fabian Kuhn

Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $n_1, \dots, n_k < n$ $(k \geq 2)$.

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

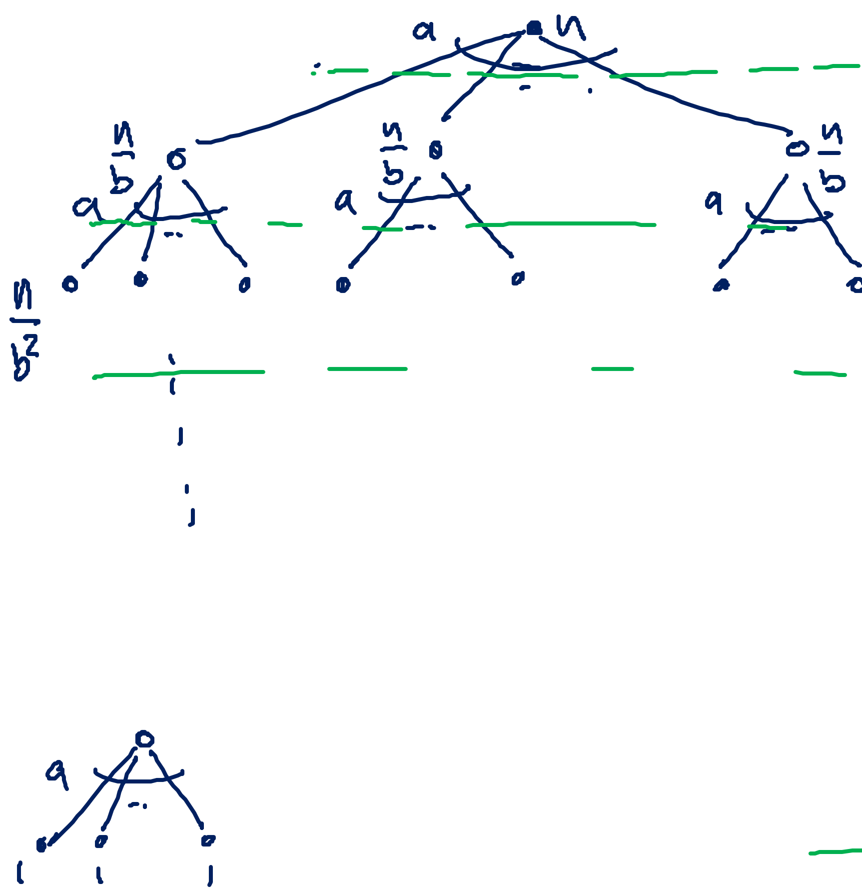
Recurrence Relations

$\log(a^{\log_b n}) = \frac{\log n}{\log b} \cdot \log a$

Recurrence relation

cost for divide & combine

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^c), \qquad T(n) = O(1) \text{ for } n \le n_0$$



n^c

$a \cdot \left(\frac{n}{b}\right)^c = \frac{a}{b^c} \cdot n^c$

$a^2 \left(\frac{n}{b^2}\right)^c = \left(\frac{a}{b^c}\right)^2 \cdot n^c$

$\vdots$

$a^{\log_b n} = n^{\log_b a}$

if $a < b^c$ $(\log_b a < c)$: $T(n) = O(n^c)$

if $\log_b a > c$: $T(n) = O(n^{\log_b a})$

if $\log_b a = c$: $T(n) = n^c \cdot \log n$

Recurrence Relations: Master Theorem

Recurrence relation

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underline{f(n)}, \qquad T(n) = O(1) \text{ for } n \le n_0$$

Cases

- $\underline{f(n) = O(n^c)}$, $\underline{c < \log_b a}$

  $$\underline{\boldsymbol{T(n) = \Theta\left(n^{\log_b a}\right)}}$$

- $f(n) = \underline{\Omega(n^c)}$, $\underline{c > \log_b a}$

  $$\underline{\boldsymbol{T(n) = \Theta\left(f(n)\right)}}$$

- $f(n) = \Theta\left(n^c \cdot \underline{\log^k n}\right)$, $k \ge 0, c = \log_b a$

  $$\boldsymbol{T(n) = \underline{\Theta\left(n^c \cdot \log^{k+1} n\right)}}$$

$T(n) = 2 \cdot T(n/2) + O(n)$

$T(n) = O(n \cdot \log n)$

Polynomials

Real polynomial $\boldsymbol{p}$ in one variable $\boldsymbol{x}$: $a_i \in \mathbb{R}$

$$p(x) = a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0$$

Coefficients of p: $a_0, a_1, \ldots, a_n \in \mathbb{R}$

Degree of p: largest power of x in p ($n - 1$ in the above case)

Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

$a_0 = 0,\ a_1 = 18,\ a_2 = -15,\ a_3 = 3$

Set of all real-valued polynomials in x: $\underline{\underline{\mathbb{R}[x]}}$ (polynomial ring)

Operations: Evaluation

- Given: Polynomial $p \in \mathbb{R}[x]$ of degree $n-1$

$$p(x) = \underline{a_{n-1}x^{n-1}} + \underline{a_{n-2}x^{n-2}} + \cdots + a_1 x + a_0$$

given some value $x_0 \in \mathbb{R}$,

compute $p(x_0)$

Operations: Evaluation

- Given: Polynomial $p \in \mathbb{R}[x]$ of degree $n-1$

$$p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

- **Horner's method** for evaluation at specific value x_0:

$$p(x_0) = \Big(\ldots\big(\big(\underline{a_{n-1}} \cdot x_0 + \underline{a_{n-2}}\big) \cdot x_0 + a_{n-3}\big) \cdot x_0 + \cdots + a_1\Big) \cdot x_0 + a_0$$

- Pseudo-code:

$p \coloneqq a_{n-1}$; $i \coloneqq n-1$;
while $(i > 0)$ **do**
    $i \coloneqq i - 1$;
    $p \coloneqq p \cdot x_0 + a_i$

- Running time: $O(n)$

Operations: Addition

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree $n-1$

$$p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$
$$q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$$

- Compute sum $p(x) + q(x)$:

$$\begin{aligned} p(x) + q(x) & \\ &= (a_{n-1}x^{n-1} + \cdots + a_0) + (b_{n-1}x^{n-1} + \cdots + b_0) \\ &= (\underline{a_{n-1} + b_{n-1}})x^{n-1} + \cdots + (\underline{a_1 + b_1})x + (\underline{a_0 + b_0}) \end{aligned}$$

$O(n)$ time

Operations: Multiplication

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree $n-1$

$$p(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

$c_0 = a_0 \cdot b_0, \; c_1 = a_1 \cdot b_0 + a_0 \cdot b_1, \; c_2 = a_0 \cdot b_2 + a_1 b_1 + a_2 b_0$

- Product $p(x) \cdot q(x)$:

$$p(x) \cdot q(x) = (a_{n-1}x^{n-1} + \cdots + a_0) \cdot (b_{n-1}x^{n-1} + \cdots + b_0)$$
$$= c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1 x + c_0$$

- Obtaining c_i: what products of monomials have degree i?

$$\text{For } 0 \leq i \leq 2n-2\text{: } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

where $a_i = b_i = 0$ for $i \geq n$.

- Running time naïve algorithm: $O(n^2)$

Faster Multiplication? $(a_{n-1}, a_{n-2}, \dots, a_{\frac{n}{2}}, a_{\frac{n}{2}-1}, \dots, a_0)$

- Multiplication is slow $\left(\Theta(n^2)\right)$
- Try divide-and-conquer to get a faster algorithm
- Assume: degree is $n-1$, n is even (n is a power of 2)
- Divide polynomial $p(x) = a_{n-1}x^{n-1} + \dots + a_0$ into 2 polynomials of degree $n/2 - 1$:

$$p_0(x) = a_{n/2-1}x^{n/2-1} + \dots + a_0$$
$$p_1(x) = a_{n-1}x^{n/2-1} + \dots + a_{n/2}$$
$$p(x) = p_1(x) \cdot x^{n/2} + p_0(x)$$

- Similarly: $q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

Use Divide-And-Conquer

- **Divide:**

$$p(x) = p_1(x) \cdot x^{n/2} + p_0(x), \qquad q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$$

- **Multiplication:**

$$p(x)q(x) = \underline{p_1(x)q_1(x)} \cdot x^n + \left(\underline{p_0(x)q_1(x)} + \underline{p_1(x)q_0(x)}\right) \cdot x^{n/2} + \underline{p_0(x)q_0(x)}$$

- 4 multiplications of degree $n/2 - 1$ polynomials:

$$\underline{T(n)} = 4\underline{T(n/2)} + \underline{O(n)}$$

Master thm:

$a=4, b=2, \log_b a = 2 > 1$

$\rightarrow T(n) = n^{\log_b a} = O(n^2)$

- Leads to $T(n) = \Theta(n^2)$ like the naive algorithm…
  - follows immediately by using the master theorem

More Clever Recursive Solution

- **Recall that**

$$p(x)q(x) = \overbrace{p_1(x)q_1(x)}^{A} \cdot x^n + \underbrace{(p_0(x)q_1(x) + p_1(x)q_0(x))}_{B} \cdot x^{n/2} + \underbrace{p_0(x)q_0(x)}_{C}$$

- Compute $r(x) = \underbrace{(p_0(x) + p_1(x))}_{\text{poly. of size } \frac{n}{2}} \cdot \underbrace{(q_0(x) + q_1(x))}_{\text{size } \frac{n}{2}}$:

$$r(x) = \underbrace{p_0(x) \cdot q_0(x)}_{C} + \underbrace{p_0(x) q_1(x) + p_1(x) q_0(x)}_{B} + \underbrace{p_1(x) \cdot q_1(x)}_{A} = A + B + C$$

compute: $r(x)$, A, C

$B = r(x) - A - C$

$$T(n) = 3 \cdot T(n/2) + O(n)$$

Karatsuba Algorithm

- Recursive multiplication:

$$r(x) = \big(p_0(x) + p_1(x)\big) \cdot \big(q_0(x) + q_1(x)\big)$$

$$\begin{aligned} p(x)q(x) = {} & p_1(x)q_1(x) \cdot x^n \\ & + \big(r(x) - p_0(x)q_0(x) + p_1(x)q_1(x)\big) \cdot x^{n/2} \\ & + p_0(x)q_0(x) \end{aligned}$$

- Recursively do 3 multiplications of degr. $(n/2 - 1)$-polynomials

$$T(n) = 3T(n/2) + O(n)$$

- Gives: $T(n) = O(n^{1.58496\ldots})$ (see Master theorem)

$\log_2 3$

Representation of Polynomials

Coefficient representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree $n-1$ is given by its n coefficients $a_0, \dots, a_{n-1}$:

$$p(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

- Coefficient vector $\boldsymbol{a} = (a_0, a_1, \dots a_{n-1})$

- Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

- The most typical (and probably most natural) representation of polynomials

Representation of Polynomials

Product of linear factors:

- Polynomial $p(x) \in \mathbb{C}[x]$ of degr. $n-1$ is given by its $n-1$ roots

$$p(x) = \underline{a_{n-1}} \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_{n-1})$$

- Example:

$$p(x) = 3x(x-2)(x-3)$$

- Every polynomial has exactly $\underline{n-1}$ roots $x_i \in \mathbb{C}$ (s.t. $p(x_i) = 0$)
  - Polynomial is uniquely defined by the $n-1$ roots and a_{n-1}

- We will not use this representation…

Representation of Polynomials

Point-value representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree $n-1$ is given by n point-value pairs:

$$p = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_{n-1}, p(x_{n-1}))\}$$

  where $x_i \neq x_j$ for $i \neq j$.

- Example: The polynomial

$$p(x) = 3x(x-2)(x-3)$$

  is uniquely defined by the four point-value pairs $(0,0), (1,6), (2,0), (3,0)$.

Operations: Coefficient Representation

$$\boldsymbol{p(x) = a_{n-1}x^{n-1} + \cdots + a_0}, \qquad \boldsymbol{q(x) = b_{n-1}x^{n-1} + \cdots + b_0}$$

Evaluation: Horner's method: Time $O(n)$

Addition:

$$p(x) + q(x) = (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0)$$

- Time: $O(n)$

Multiplication:

$$p(x) \cdot q(x) = c_{2n-2}x^{2n-2} + \cdots + c_0, \qquad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

- Naive solution: Need to compute product $a_i b_j$ for all $0 \leq i, j \leq n$
- Time: $O(n^2)$ → can be improved to $O(n^{1.58...})$

Operations: Linear Factors (Roots)

$$p(x) = a_{n-1} \cdot (x - x_1) \cdot \ldots \cdot (x - x_{n-1})$$
$$q(x) = b_{n-1} \cdot (x - x_1) \cdot \ldots \cdot (x - x_{n-1})$$

Evaluation:

- Just plug in the value where the poly. is evaluated: Time $O(n)$

Multiplication:

- Concatenate the two representations: Time $O(n)$

Addition:

- Need to find the roots of $p(x) + q(x)$
- For polynomials of degree > 4, this is not possible by using basic arithmetic operations ($+, -, \cdot, /, \sqrt[a]{b}$)
- In the usual computational model impossible
  – Numerically, the roots can be computed to arbitrary precision

Operations: Point-Value Representation

$$\boldsymbol{p = \{(x_0, p(x_0)), \dots, (x_{n-1}, p(x_{n-1}))\}}$$
$$\boldsymbol{q = \{(x_0, q(x_0)), \dots, (x_{n-1}, q(x_{n-1}))\}}$$

- Note: we use the **same points $\boldsymbol{x_0, \dots, x_{n-1}}$** for both polynomials

Addition:

$$p + q = \{(x_0, p(x_0) + q(x_0)), \dots, (x_{n-1}, p(x_{n-1}) + q(x_{n-1}))\}$$

- Time: $O(n)$

Multiplication:

$$p \cdot q = \{(x_0, p(x_0) \cdot q(x_0)), \dots, (x_{2n-2}, p(x_{2n-2}) \cdot q(x_{2n-2}))\}$$

- Time: $O(n)$

Evaluation: Polynomial interpolation can be done in $O(n^2)$

Operations on Polynomials

Cost depending on representation:

	Coefficient	~~Roots~~	Point-Value
Evaluation	$\boldsymbol{O(n)}$	$\boldsymbol{O(n)}$	$\boldsymbol{O(n^2)}$
Addition	$\boldsymbol{O(n)}$	∞	$\boldsymbol{O(n)}$
Multiplication	$\boldsymbol{O(n^{1.58})}$	$\boldsymbol{O(n)}$	$\boldsymbol{O(n)}$

Faster Polynomial Multiplication?

Multiplication is fast when using the point-value representation

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients
↓ **Evaluation** at points $x_0, x_1, \dots, x_{2n-2}$
$2 \times 2n$ point-value pairs $\big(x_i, p(x_i)\big)$ and $\big(x_i, q(x_i)\big)$
↓ **Point-wise multiplication** $O(n)$ time
$2n$ point-value pairs $\big(x_i, p(x_i)q(x_i)\big)$
↓ **Interpolation**
$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Coefficients to Point-Value Representation

Given: Polynomial $p(x)$ by the coefficient vector $(a_0, a_1, \dots, a_{N-1})$

Goal: Compute $p(x)$ for all x in a given set X

- Where X is of size $|X| = N$
- Assume that N is a power of 2

Divide and Conquer Approach

- Divide $p(x)$ of degree $N - 1$ (N is even) into 2 polynomials of degree $N/2 - 1$ differently than in Karatsuba's algorithm

- $p_0(y) = a_0 + a_2 y + a_4 y^2 + \dots + a_{N-2} y^{N/2-1}$ (even coeff.)

  $p_1(y) = a_1 + a_3 y + a_5 y^2 + \dots + a_{N-1} y^{N/2-1}$ (odd coeff.)

Coefficients to Point-Value Representation

Goal: Compute $p(x)$ for all x in a given set X of size $|X| = N$

- Divide $p(x)$ of degr. $N-1$ into 2 polynomials of degr. $N/2 - 1$

$$p_0(y) = a_0 + a_2 y + a_4 y^2 + \cdots + a_{N-2} y^{N/2-1} \quad \text{(even coeff.)}$$
$$p_1(y) = a_1 + a_3 y + a_5 y^2 + \cdots + a_{N-1} y^{N/2-1} \quad \text{(odd coeff.)}$$

Let's first look at the "combine" step:

- We need to compute $p(x)$ for all $x \in X$ after recursive calls for polynomials p_0 and p_1:

$\forall x \in X: \quad p(x) = p_0(x^2) + x \cdot p_1(x^2)$

recursively compute $p_0(y), p_1(y)$ for all $y \in X^2$

$X^2 := \{x^2 : x \in X\}$

cost of divide & combine: $O(N + |X|)$

Coefficients to Point-Value Representation

Goal: Compute $p(x)$ for all x in a given set X of size $|X| = N$

- Divide $p(x)$ of degr. $N-1$ into 2 polynomials of degr. $N/2 - 1$

$$p_0(y) = a_0 + a_2 y + a_4 y^2 + \cdots + a_{N-2} y^{N/2-1} \quad \text{(even coeff.)}$$
$$p_1(y) = a_1 + a_3 y + a_5 y^2 + \cdots + a_{N-1} y^{N/2-1} \quad \text{(odd coeff.)}$$

Let's first look at the "combine" step:

$$\forall x \in X: \quad p(x) = p_0(x^2) + x \cdot p_1(x^2)$$

- Recursively compute $p_0(y)$ and $p_1(y)$ for all $y \in X^2$
  - Where $X^2 := \{x^2 : x \in X\}$
- Generally, we have $|X^2| = |X|$

Analysis

$|X| = n$

Recurrence formula for the given algorithm:

$$T(n,|X|) = 2T\left(\frac{n}{2}, |X^2|\right) + O(n+|X|) \le 2T\left(\frac{n}{2}, |X|\right) + O(n+|X|)$$

$n, |X|$ $\rightarrow$ $n + |X|$

$\frac{n}{2}, |X^2|$ $\quad$ $\frac{n}{2}, |X^2|$ $\rightarrow$ $2\cdot\left(\frac{n}{2} + |X^2|\right) = n + 2|X^2|$

$\rightarrow$ $4\left(\frac{n}{4} + |X^4|\right) = n + 4|X^4|$

$\vdots$

$n(1 + |X^n|) = n + n|X^n| \le O(n^2)$

$|X^2| \overset{?}{=} |X|$

$\{-1, 1\}$

if $|X^{2k}| = \dfrac{|X^k|}{2}$,

we would get an $O(n \log n)$ alg.

Faster Algorithm?

- In order to have a faster algorithm, we need $|X^2| < |X|$

$n \to \frac{n}{2} \to \frac{n}{4}$

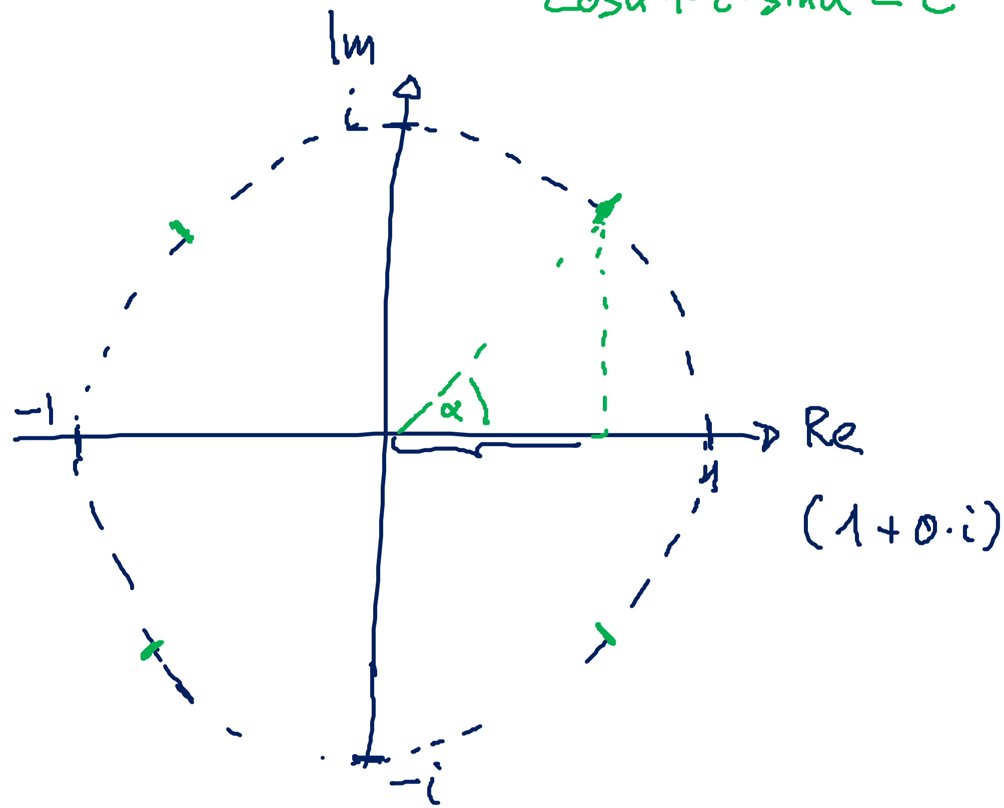
$\cos\alpha + i\cdot\sin\alpha = e^{i\alpha}$

$\{1\}$

$\{-1, 1\}$

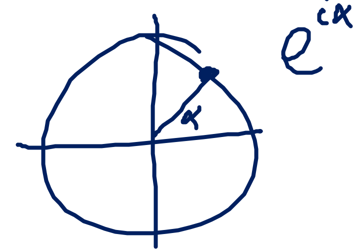
$\{i, -i, -1, 1\}$

$\{1, -1, i, -i, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$
$-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$



Choice of X

- Select points $x_0, x_1, \dots, x_{N-1}$ to evaluate p and q in a clever way



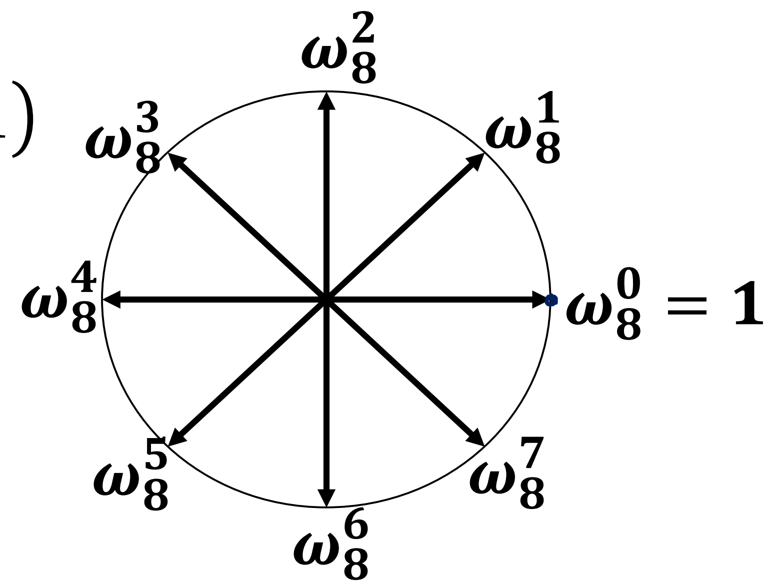
Consider the N complex roots of unity:

Principle root of unity: $\omega_N = e^{2\pi i/N}$

$$\left(i = \sqrt{-1}, \qquad e^{2\pi i} = 1\right)$$

Powers of ω_N (roots of unity):

$$1 = \omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}$$



Note: $\omega_N^k = e^{2\pi i k/N} = \cos\frac{2\pi k}{N} + i \cdot \sin\frac{2\pi k}{N}$

- **Cancellation Lemma:**

  For all integers $n > 0$, $k \geq 0$, and $d > 0$, we have:

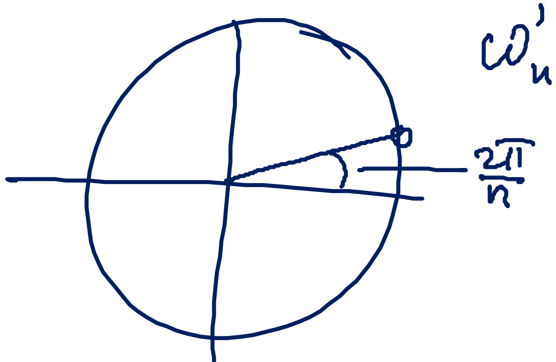
$$\boldsymbol{\omega_{dn}^{dk} = \omega_n^k}, \qquad \boldsymbol{\omega_n^{k+n} = \omega_n^k}$$

- **Proof:**

$$\omega_n^k = e^{i \cdot \frac{2\pi}{n} \cdot k}$$

$$\omega_{dn}^{dk} = e^{i \frac{2\pi}{dn} \cdot dk} = \omega_n^k$$

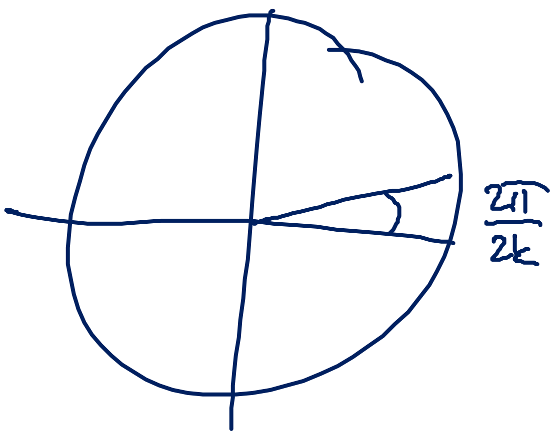
$$\omega_n^{k+n} = e^{i \frac{2\pi}{n} k} \cdot \underbrace{e^{i \frac{2\pi}{n} \cdot n}}_{e^{2\pi i} = 1} = \omega_n^k$$



Properties of the Roots of Unity

Claim: If $X = \left\{\omega_{2k}^{i} : i \in \{0, \dots, 2k-1\}\right\}$, we have

$$\boldsymbol{X^2 = \left\{\omega_k^i : i \in \{0, \dots, k-1\}\right\}}, \qquad \boldsymbol{|X^2| = \frac{|X|}{2}}$$



$$X^2 = \left\{\omega_{2k}^{2i} = \omega_k^i,\ i \in \{0, \dots, 2k-1\}\right\}$$

$$= \left\{\omega_k^i,\ i \in \{0, \dots, k-1\}\right\}$$

Analysis

New recurrence formula:

$$T(N, |X|) \leq 2 \cdot T\left({}^{N}\!/_{2}, {}^{|X|}\!/_{2}\right) + O(N + |X|)$$

for $|X| = N$

$\to \Theta(N \log N)$

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n - 1$, n coefficients

FFT **Evaluation** at points $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$ $O(n \log n)$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

Point-wise multiplication

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)\right)$

Interpolation

$p(x)q(x)$ of degree $2n - 2$, $2n - 1$ coefficients