Chapter 3
Dynamic Programming

Algorithm Theory
WS 2019/20

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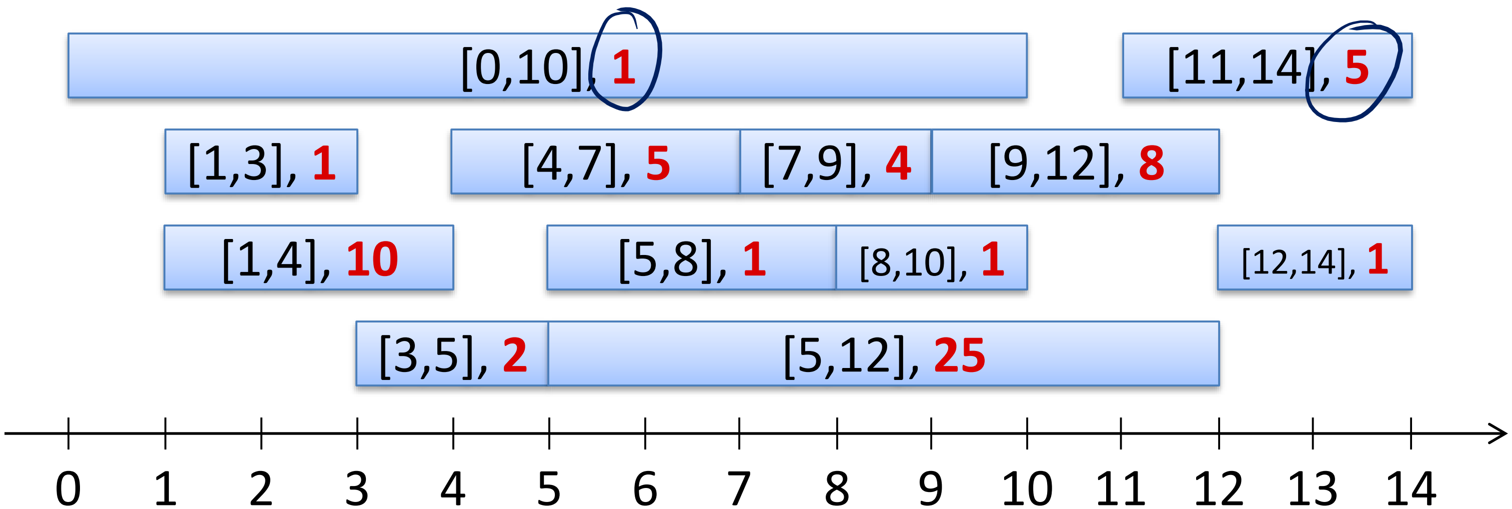
Weighted Interval Scheduling

- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]
- Each interval has a **weight** w

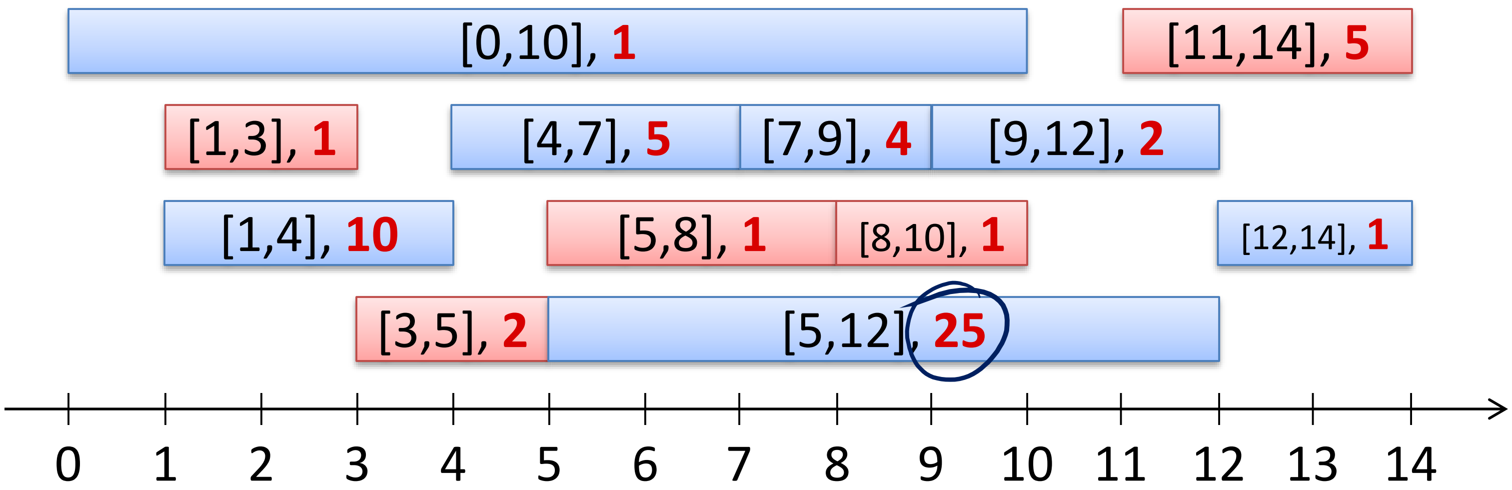


- **Goal:** Non-overlapping set of intervals of largest possible weight
  - Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping

- **Example:** Intervals are room requests of different importance

Greedy Algorithms

Choose available request with earliest finishing time:



- Algorithm is not optimal any more
  - It can even be arbitrarily bad…

- No greedy algorithm known that works

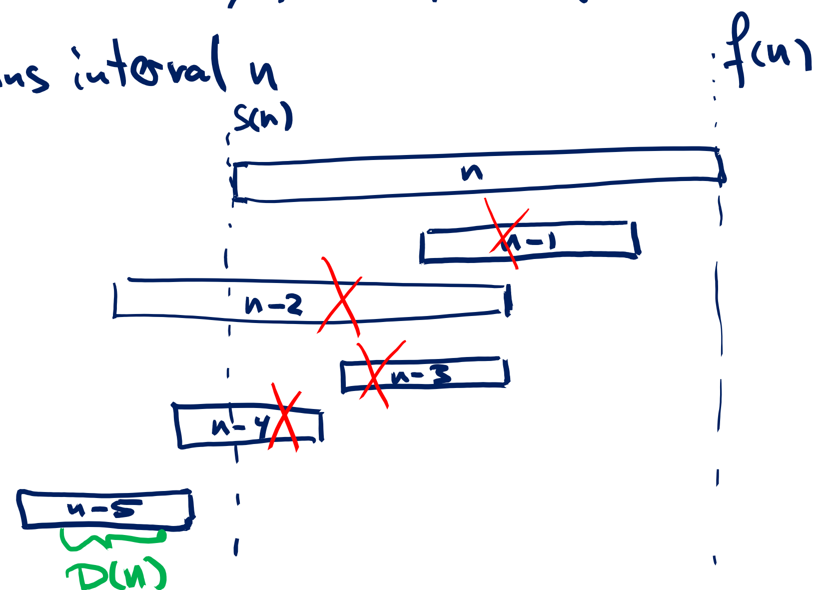
Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$ > 0
- Assume intervals $1, \dots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$
- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval n

case 1: opt. solution does not contain interval n

$\Longrightarrow$ opt. sol. for all intervals $1, \dots, n$ = opt. sol. for intervals $1, \dots, n-1$

case 2: opt. solution contains interval n



in example:

opt. solution consists of interval n

+

opt. solution of intervals $1, \dots, \underbrace{n-5}_{p(n)}$

Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$
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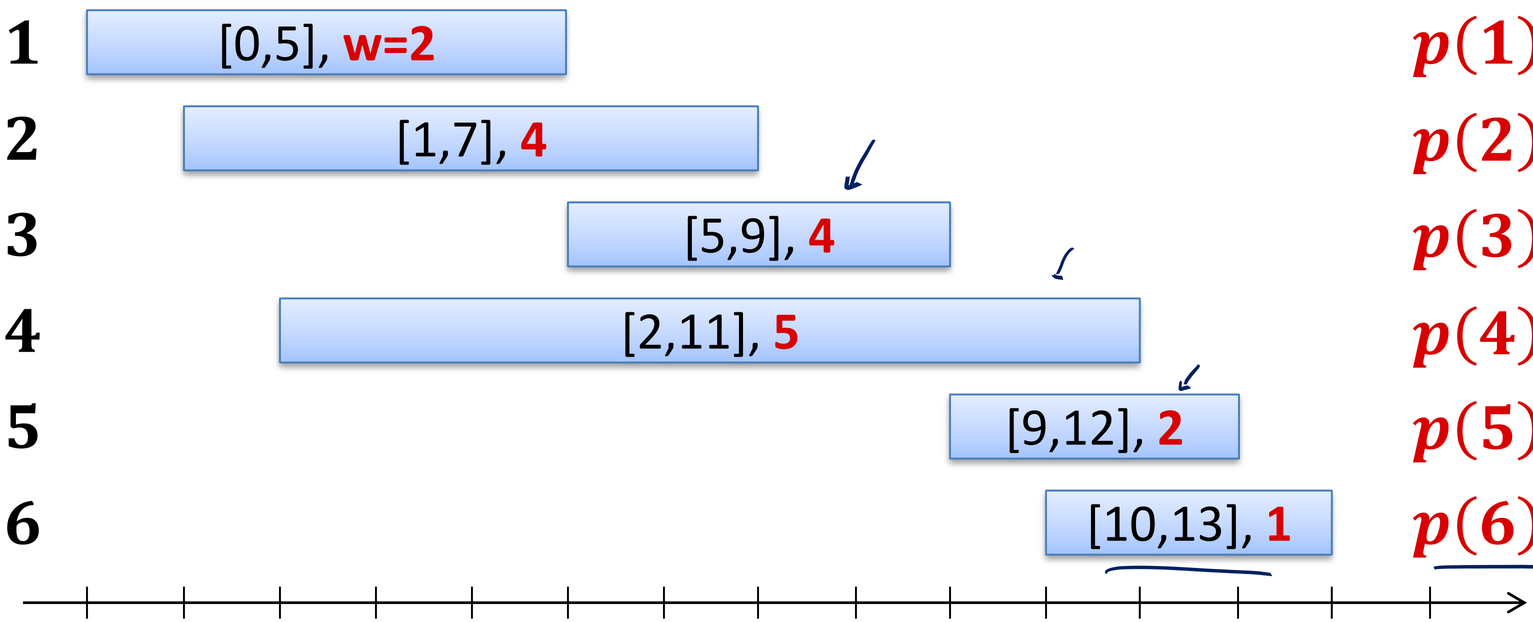
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$
- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval n
- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$
  Define $p(k) := \max\{i \in \{0, \dots, k-1\} : f(i) \leq s(k)\}$

$$W(k) = \max\{\underbrace{W(k-1)}_{\text{if opt. sol. does contain } k}, \underbrace{W(p(k)) + w(k)}_{\text{opt. sol. contains } k}\}$$

- Opt. solution does not contain interval n: $\mathbf{W(n) = W(n-1)}$
  Opt. solution contains interval n: $\mathbf{W(n) = w(n) + W(p(n))}$

Example

Interval:



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

compute $p(k)$:

using binary search : $O(\log k)$

all k: $O(n \log n)$

Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \dots, k$
  - $p(k)$: last interval to finish before interval k starts

- Recursive definition of optimal weight:

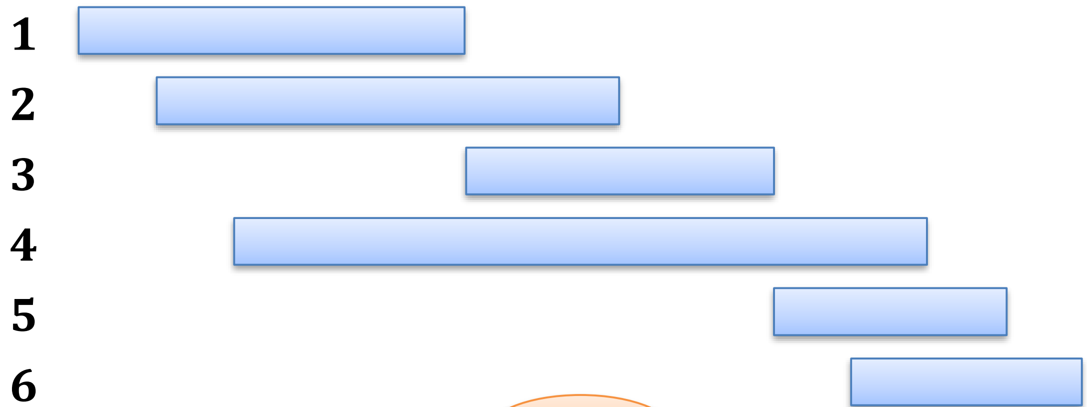
$$\forall k > 1: \; W(k) = \max\{W(k-1), w(k) + W(p(k))\}$$

$$W(1) = w(1)$$

Immediately gives a simple, recursive algorithm

```
Compute p(k) values for all k
W(k):
    if k == 1:
        x = w(1)
    else:
        x = max{W(k-1), w(k) + W(p(k))}
    return x
```

Running Time of Recursive Algorithm



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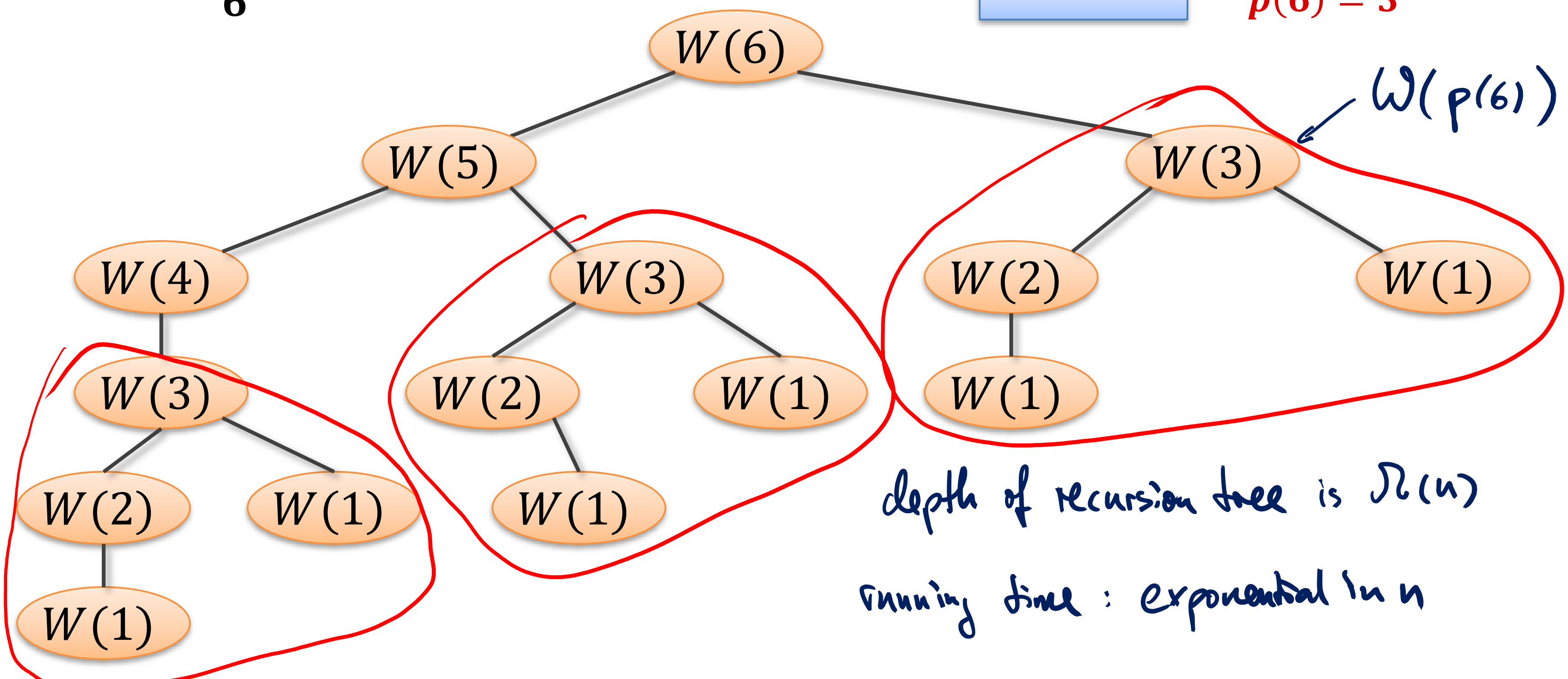


depth of recursion tree is $\Omega(n)$

running time : exponential in n

Memoizing the Recursion

- Running time of recursive algorithm: exponential!
- But, alg. only solves n different sub-problems: $W(1), \dots, W(n)$
- There is no need to compute them multiple times

Memoization: Store already computed values for future rec. calls

```
Compute p(k) for all k
memo = {};        hash table
W(k):
    if k in memo: return memo[k]
    if k == 1:
        x = w(1)
    else:
        x = max{W(k-1), w(k) + W(p(k))}
    memo[k] = x
    return x
```

Dynamic Programming (DP)

DP ≈ Recursion + Memoization

Recursion: Express problem *recursively* in terms of (a 'small' number of) *subproblems* (of the same kind)

Memoize: *Store* solutions for *subproblems* reuse the stored solutions if the same subproblems has to be solved again

Weighted interval scheduling: subproblems $W(1), W(2), W(3), \ldots$

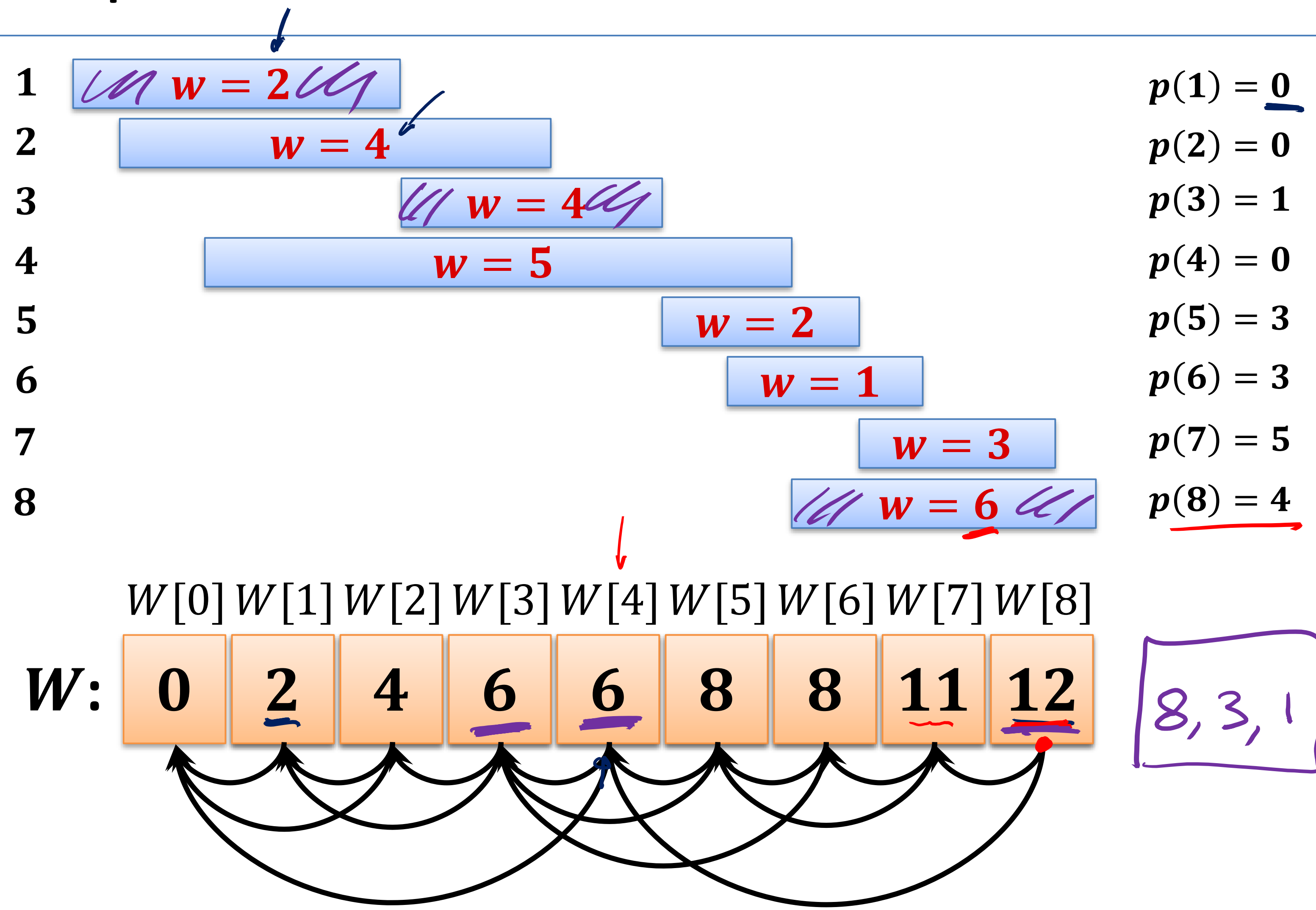
$$\text{runtime} = \underbrace{\#\text{subproblems}}_{n \text{ in case of int. scheduling}} \cdot \underbrace{\text{time per subproblem}}_{\#\text{ of subproblems a subproblem depends on}}$$

DP: Some History …

- Where das does the name come from?
- DP was developed by Richard E. Bellman in 1940s/1950s.
- In his autobiography, it says:

  "I spent the Fall quarter (of 1950) at RAND. My first task was to find a name for multistage decision processes. … The 1950s were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word research. … His face would suffuse, he would turn red, and he would get violent if people used the term research in his presence. You can imagine how he felt, then, about the term mathematical. … Hence, I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. What title, what name, could I choose? In the first place I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons. I decided therefore to use the word "programming". I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying. … It also has a very interesting property as an adjective, and that it's impossible to use the the word dynamic in a pejorative sense. … Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. …"

Example



Computing the schedule: **store where you come from!**

Dynamic Programming

„Memoization“ for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)
- Number of sub-problems that need to be considered is small

Matrix-chain multiplication

$A_i : d_{i-1} \times d_i$

Given: sequence (chain) $\langle A_1, A_2, \dots, A_n \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \dots \cdot A_n$

$(A_1 \cdots A_{n/2}) \cdot (A_{n/2+1} \cdots A_n)$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

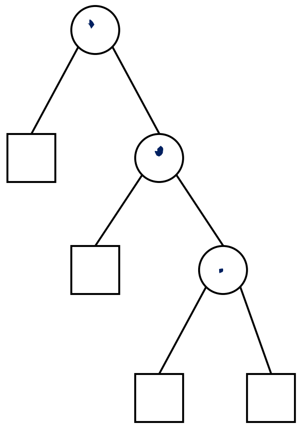
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

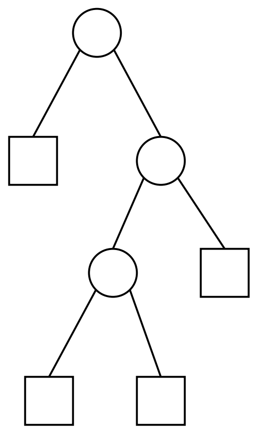
$$(((A_1A_2)A_3)A_4)$$

Different parenthesizations

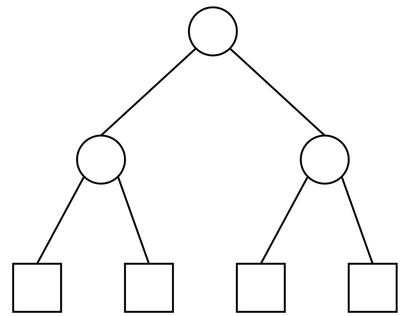
Different parenthesizations correspond to different trees:



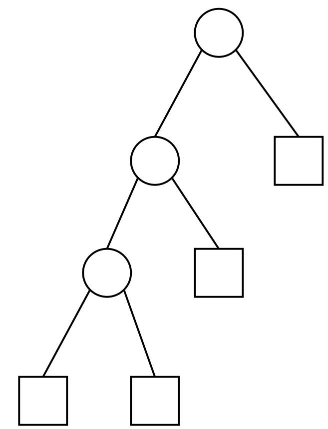
$$\big(A_1(A_2(A_3A_4))\big)$$



$$\big(A_1((A_2A_3)A_4)\big)$$



$$\big((A_1A_2)(A_3A_4)\big)$$



$$\big(((A_1A_2)A_3)A_4\big)$$

Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \qquad \text{for } n \geq 2$$

exponential

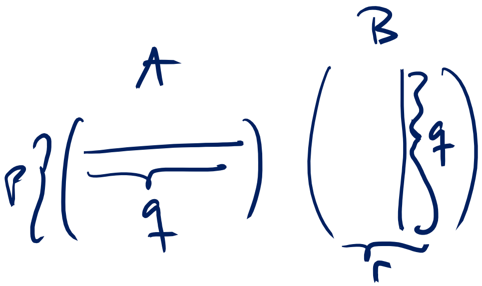
$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

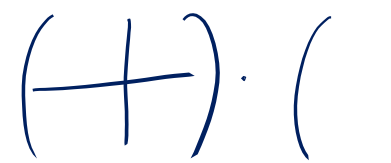
- Thus: Exhaustive search needs exponential time!

Multiplying Two Matrices

$$A = (a_{ij})_{p\times q}, \qquad B = (b_{ij})_{q\times r}, \qquad A \cdot B = C = (c_{ij})_{p\times r}$$



$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$



Algorithm *Matrix-Mult*

Input: $(p \times q)$ matrix A, $(q \times r)$ matrix B

Output: $(p \times r)$ matrix $C = A \cdot B$

```
for i := 1 to p do
    for j := 1 to r do
        C[i,j] := 0;
        for k := 1 to q do
            C[i,j] := C[i,j] + A[i,k] · B[k,j]
```

Remark:

Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.373})$ multiplications.

by divide & conquer

Number of multiplications and additions: $\boldsymbol{p \cdot q \cdot r}$

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (50×5) matrix

A_2 : (5×100) matrix

A_3 : (100×10) matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $\big(A_1(A_2 A_3)\big)$ require:

5×10

$A' = (A_1 A_2)$: $50 \cdot 5 \cdot 100 = 25'000$ $A'' = (A_2 A_3)$: $5 \cdot 100 \cdot 10 = 5000$

50×100

$A' A_3$: $50 \cdot 100 \cdot 10 = 50'000$ $A_1 A''$: $50 \cdot 5 \cdot 10 = 2500$

Sum: $75'000$ $7'500$

Structure of an Optimal Parenthesization

- $(A_{\ell\ldots r})$: optimal parenthesization of $A_\ell \cdot \ldots \cdot A_r$

  For some $1 \leq k < n$: $\boldsymbol{(A_{1\ldots n}) = ((A_{1\ldots k}) \cdot (A_{k+1\ldots n}))}$

- Any optimal solution contains optimal solutions for sub-problems

- Assume matrix A_i is a $(d_{i-1} \times d_i)$-matrix

- Cost to solve sub-problem $A_\ell \cdot \ldots \cdot A_r, \ell \leq r$ optimally: $C(\ell, r)$

- Then:

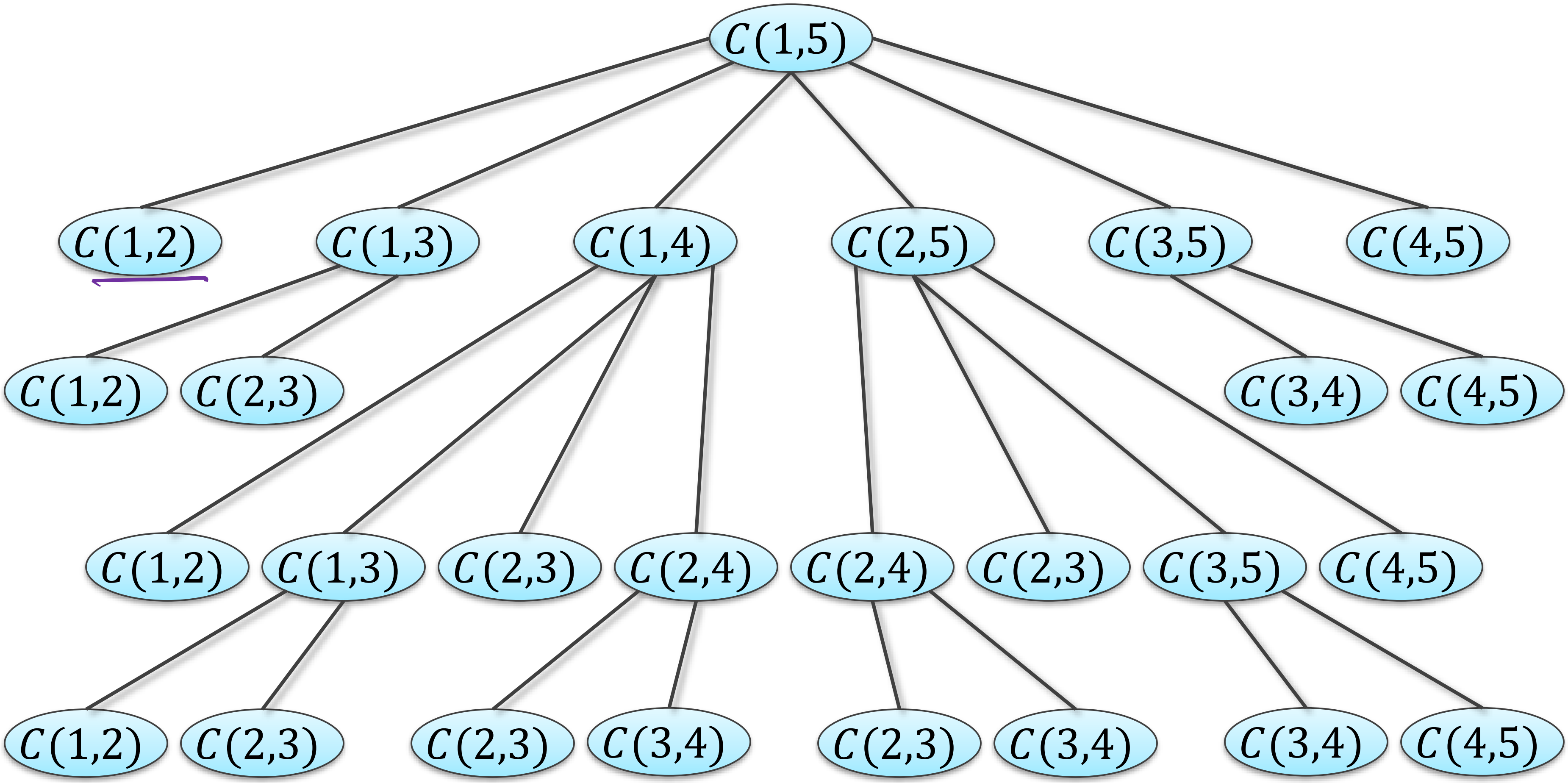
  a, . . . k . . . , b

$$\boldsymbol{C(a,b) = \min_{a \leq k < b} \{ C(a,k) + C(k+1,b) + d_{a-1} d_k d_b \}}$$

$$\boldsymbol{C(a,a) = 0}$$

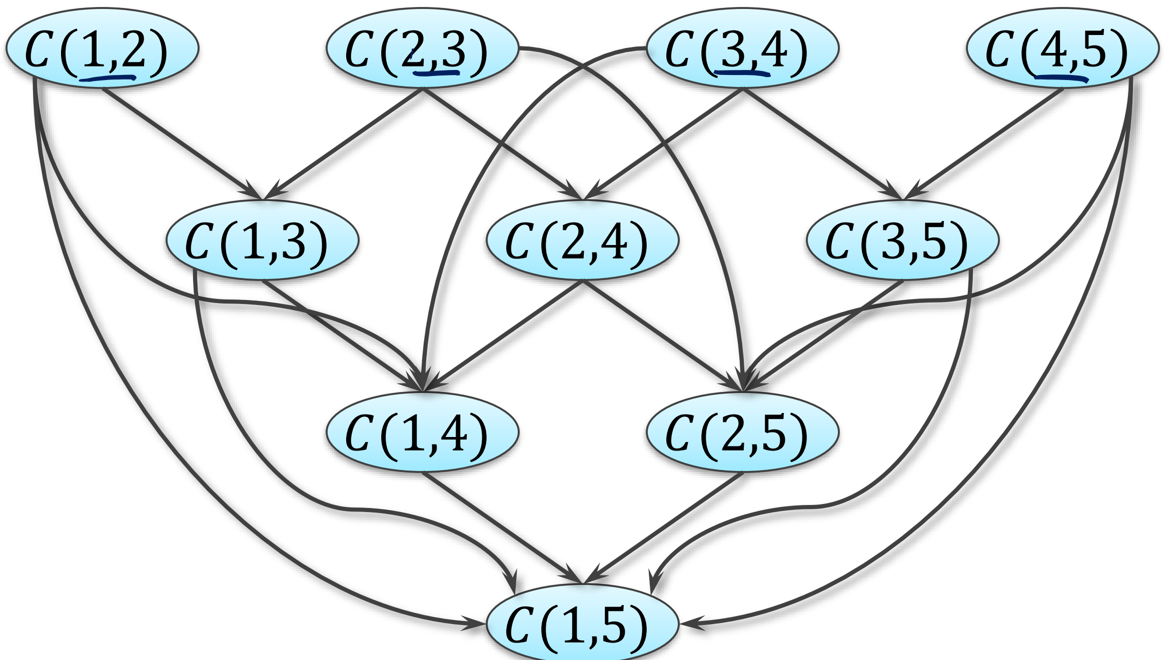
Recursive Computation of Opt. Solution

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Using Meomization

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j), i < j$ is computed exactly once → $O(n^2)$ values
- Each $C(i,j)$ dir. depends on $C(i,k), C(k,j)$ for $i < k < j$

Cost for each $C(i,j)$: $O(n)$ → overall time: $\boldsymbol{O(n^3)}$

Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time

$$O(n \cdot \log n).$$

2. There is a linear time algorithm that determines a parenthesization using at most

$$1.155 \cdot C(1, n)$$

multiplications.

Knapsack

- n items $1, \dots, n$, each item has weight w_i and value v_i
- Knapsack (bag) of capacity W

- Goal: pack items into knapsack such that total weight is at most W and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t.} \quad S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length w_i and value v_i, server available for W time units, try to execute a set of jobs that maximizes the total value

Recursive Structure?

OPT(k): opt. value when only items 1, …, k are available

- Optimal solution: $\mathcal{O}$
- If $n \notin \mathcal{O}$: $\mathrm{OPT}(n) = \mathrm{OPT}(n-1)$
- What if $n \in \mathcal{O}$?
  - Taking n gives value v_n
  - But, n also occupies space w_n in the bag (knapsack)
  - There is space for $W - w_n$ total weight left!

$$\mathrm{OPT}(n) = v_n + \underbrace{\text{optimal solution with first } n-1 \text{ items and knapsack of capacity } W - w_n}_{\text{not just } \mathrm{OPT}(n-1)}$$

A More Complicated Recursion

end goal: $OPT(n, W)$

$\mathbf{OPT}(\boldsymbol{k}, \boldsymbol{x})$: value of optimal solution with items $1, \dots, k$
and knapsack of capacity x

Recursion:

$$OPT(k,x) = \max\Big\{\underbrace{OPT(k-1,x)}_{\text{opt. sol. when not using item } k},\ \overbrace{v_k + OPT(k-1,\underbrace{x-w_k}_{\text{rem. cap.}})}^{\text{only if } x \geq w_k}\Big\}$$

initialization

$OPT(0, x) = 0$

$OPT(k, 0) = 0$

subproblems?

arbitrary weights $= 2^n$ NP-hard

integer weights : $n \cdot W$

$\hookrightarrow$ time $O(n \cdot W)$

Dynamic Programming Algorithm

Set up table for all possible $\mathrm{OPT}(k, x)$-values

- Assume that all weights w_i are integers!

	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$		$\cdots$			x			W
$\mathbf{0}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{1}$	0											
$\mathbf{2}$	0											
$\mathbf{3}$	0											
$\vdots$	0											
	0											
k	0											
$\boldsymbol{n}$	0											

$\mathrm{OPT}(k, x)$

w_k

Row $\boldsymbol{i}$, column $\boldsymbol{j}$:

$\boldsymbol{OPT(i, j)}$

Example

- 8 items: (3,2), (2,4), (4,1), (5,6), (3,3), (4,3), (5, 4), (6,6)
  Knapsack capacity: 12

  weight **value**

- $\boldsymbol{OPT(k, x) = \max\{OPT(k-1, x), OPT(k-1, \overbrace{x - w_k}^{\geq 0}) + v_k\}}$

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	2	2	2	2	2	2	2	2	2	2
2	0	4	4	4	6	6	6	6	6	6	6	6
3												
4												
5												
6												
7												
8												

Running Time of Knapsack Algorithm

- **Size of table:** $O(n \cdot W)$
- Time per table entry: $O(1)$ → **overall time: $\boldsymbol{O(nW)}$**
- Computing solution (set of items to pick):
  Follow $\leq n$ arrows → $O(n)$ time (after filling table)
- Note: Time depends on W → can be exponential in n…
- And it is problematic if weights are not integers.

still possible if weights are rational ← not really reasonable

another special case: values are integers

– general case is NP-hard