Chapter 4
Amortized Analysis

Algorithm Theory
WS 2019/20

Fabian Kuhn

Amortized Cost

Amortized Cost of sequence of operations $\boldsymbol{i = 1, 2, \dots, n}$

- Actual cost of op. i: $\boldsymbol{t_i}$
- Amortized cost of op. i is $\boldsymbol{a_i}$ if for every possible seq. of ops.,

$$T = \sum_{i=1}^{n} t_i \leq \sum_{i=1}^{n} a_i$$

Amortized Analysis: Techniques

1. Directly analyze the total cost of all operations
2. Accounting method
   - Bank account with initial balance 0
   - Paying x to bank costs x
   - Use money from the bank to pay for expensive operations
3. Potential function method

Potential Function Method

- Most generic and elegant way to do amortized analysis!
  - But, also more abstract than the others…

- State of data structure / system: $S \in \mathcal{S}$ (state space)

  Potential function $\boldsymbol{\Phi}: \boldsymbol{\mathcal{S}} \to \mathbb{R}_{\geq 0}$

- **Operation $\boldsymbol{i}$:**
  - $\boldsymbol{t_i}$: actual cost of operation i
  - $\boldsymbol{S_i}$: state after execution of operation i (S_0: initial state)
  - $\boldsymbol{\Phi_i} \coloneqq \Phi(S_i)$: potential after exec. of operation i
  - $\boldsymbol{a_i}$: amortized cost of operation i:

$$\boldsymbol{a_i \coloneqq t_i + \Phi_i - \Phi_{i-1}}$$

Potential Function Method

Operation i:

actual cost: t_i amortized cost: $a_i = t_i + \Phi_i - \Phi_{i-1}$

Overall cost:

$$T := \sum_{i=1}^{n} t_i = \left(\sum_{i}^{n} a_i\right) + \Phi_0 - \Phi_n$$

≥ 0

$\sum a_i \geq \sum t_i - \Phi_0$

$\sum a_i \geq \sum t_i$ if $\Phi_0 = 0$

Example 3: Dynamic Array

- How to create an array where the size dynamically adapts to the number of elements stored?
  - e.g., Java "ArrayList" or Python "list"

Implementation:

- Initialize with initial size N_0
- Assumptions: Array can only grow by appending new elements at the end
- If array is full, the size of the array is increased by a factor $\beta > 1$

Operations (array of size N):

- read / write: actual cost $O(1)$
- append: actual cost is $O(1)$ if array is not full, otherwise the append cost is $O(\beta \cdot N)$ (new array size)

size before increasing

Example 3: Dynamic Array

Notation:

- n: number of elements stored
- N: current size of array

Cost t_i of i^{th} append operation: $t_i = \begin{cases} 1 & \text{if } n < N \\ \beta \cdot N & \text{if } n = N \end{cases}$

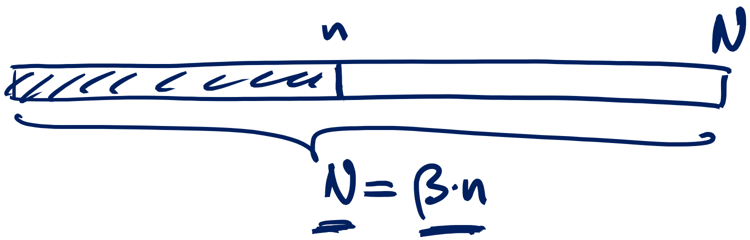
Claim: Amortized append cost is $O(1)$

Potential function Φ?

- should allow to pay expensive append operations by cheap ones
- when array is full, Φ has to be large
- immediately after increasing the size of the array, Φ should be small again

Dynamic Array: Potential Function

Cost t_i of i^{th} append operation: $t_i = \begin{cases} 1 & \text{if } n < N \\ \beta \cdot N & \text{if } n = N \end{cases}$



ϕ small ($\phi = 0$)

ϕ large ($\phi \geq \beta N$)

at the beginning

$N = N_0$

$n = 0$

$$\phi(n,N) = c\,(\beta n - N) + c N_0$$

$$c(\beta N - N) \geq \beta N$$

$$c(\beta - 1) \geq \beta$$

$$c \geq \frac{\beta}{\beta - 1}$$

$$\boxed{\phi(n,N) = \frac{\beta}{\beta-1}(\beta n - N) + \frac{\beta}{\beta-1} N_0}$$

Dynamic Array: Amortized Cost

Cost t_i of i^{th} append operation: $t_i = \begin{cases} 1 & \text{if } n < N \\ \beta \cdot N & \text{if } n = N \end{cases}$

$$\Phi(n,N) = \frac{\beta}{\beta-1}\left(\beta n - N + N_0\right)$$

---

amortized cost a_i

case 1 ($n < N$): $a_i = 1 + \frac{\beta}{\beta-1}\left(\beta(n+1) - \beta n\right) = 1 + \frac{\beta^2}{\beta-1}$

case 2 ($n = N$): $t_i = \beta n = \beta N$

$$a_i = \beta N + \frac{\beta}{\beta-1}\cdot\left[\beta(N+1) - \beta N - (\beta N - N)\right]$$

$$= \beta N + \frac{\beta^2}{\beta-1} - \frac{\beta}{\beta-1}\cdot(\beta-1)N = \frac{\beta^2}{\beta-1}$$

$$\boxed{\text{amortized cost} \le 1 + \frac{\beta^2}{\beta-1}}$$