# Chapter 5

# Data Structures

## Algorithm Theory
## WS 2019/20

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# Examples

**Dictionary:**

- Operations: insert(*key,value*), delete(*key*), find(*key*)
- Implementations:
  - Linked list: all operations take $O(n)$ time ($n$: size of data structure)
  - Balanced binary tree: all operations take $O(\log n)$ time
  - Hash table: all operations take $O(1)$ times (with some assumptions)

**Stack (LIFO Queue):**

- Operations: push, pull
- Linked list: $O(1)$ for both operations

**(FIFO) Queue:**

- Operations: enqueue, dequeue
- Linked list: $O(1)$ time for both operations

Here: **Priority Queues (heaps)**, **Union-Find data structure**

# Dijkstra's Algorithm

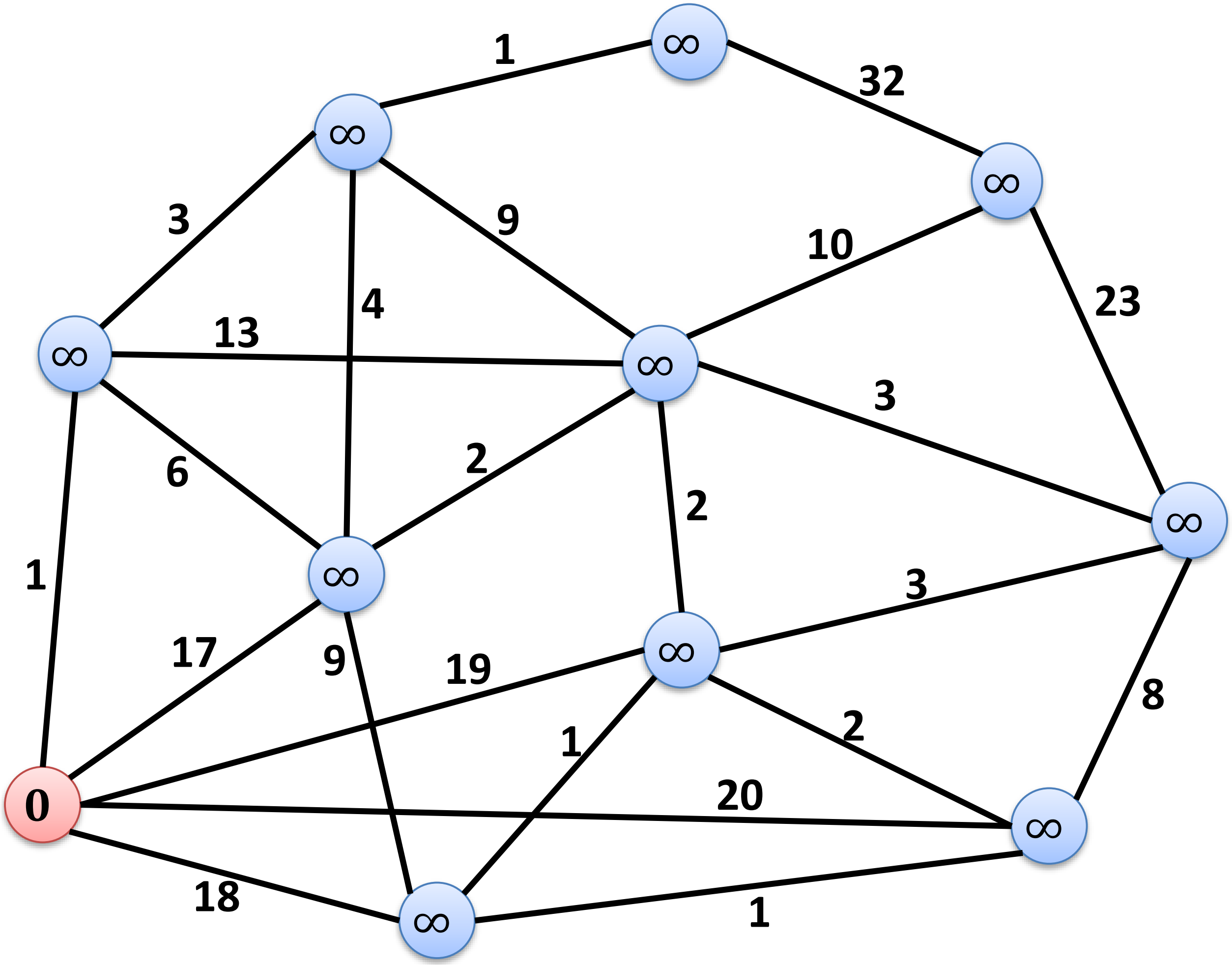
## Single-Source Shortest Path Problem:

- **Given:** graph $G = (V, E)$ with edge weights $w(e) \geq 0$ for $e \in E$
  source node $s \in V$
- **Goal:** compute shortest paths from $s$ to all $v \in V$

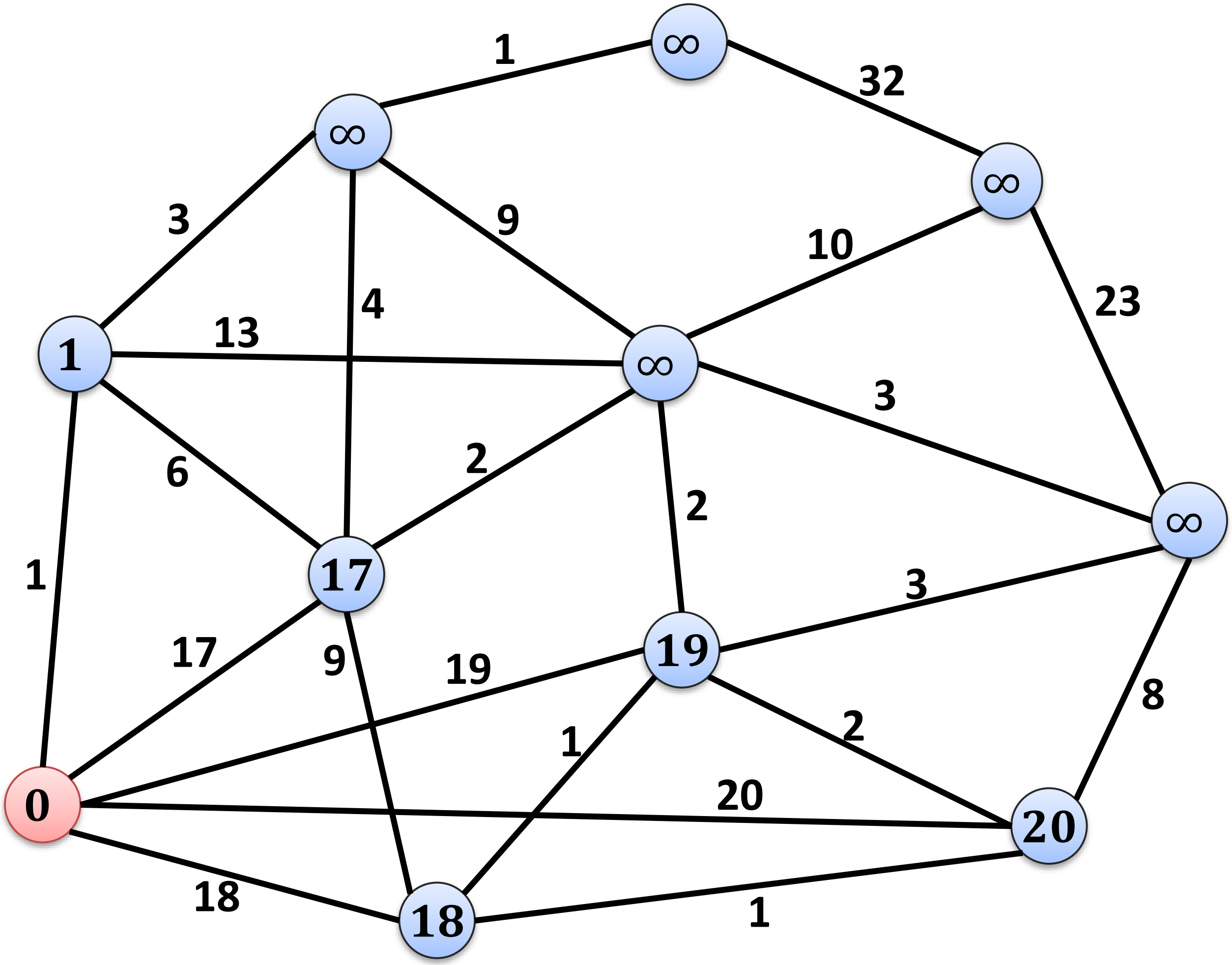
## Dijkstra's Algorithm:

```
1.  Initialize $d(s,s) = 0$ and $d(s,v) = \infty$ for all $v \neq s$
2.  All nodes are unmarked
3.  Get unmarked node $u$ which minimizes $d(s,u)$:
4.      For all $e = \{u,v\} \in E$, $d(s,v) = \min\{d(s,v), d(s,u) + w(e)\}$
5.      mark node $u$
6.  Until all nodes are marked
```

# Example



# Example



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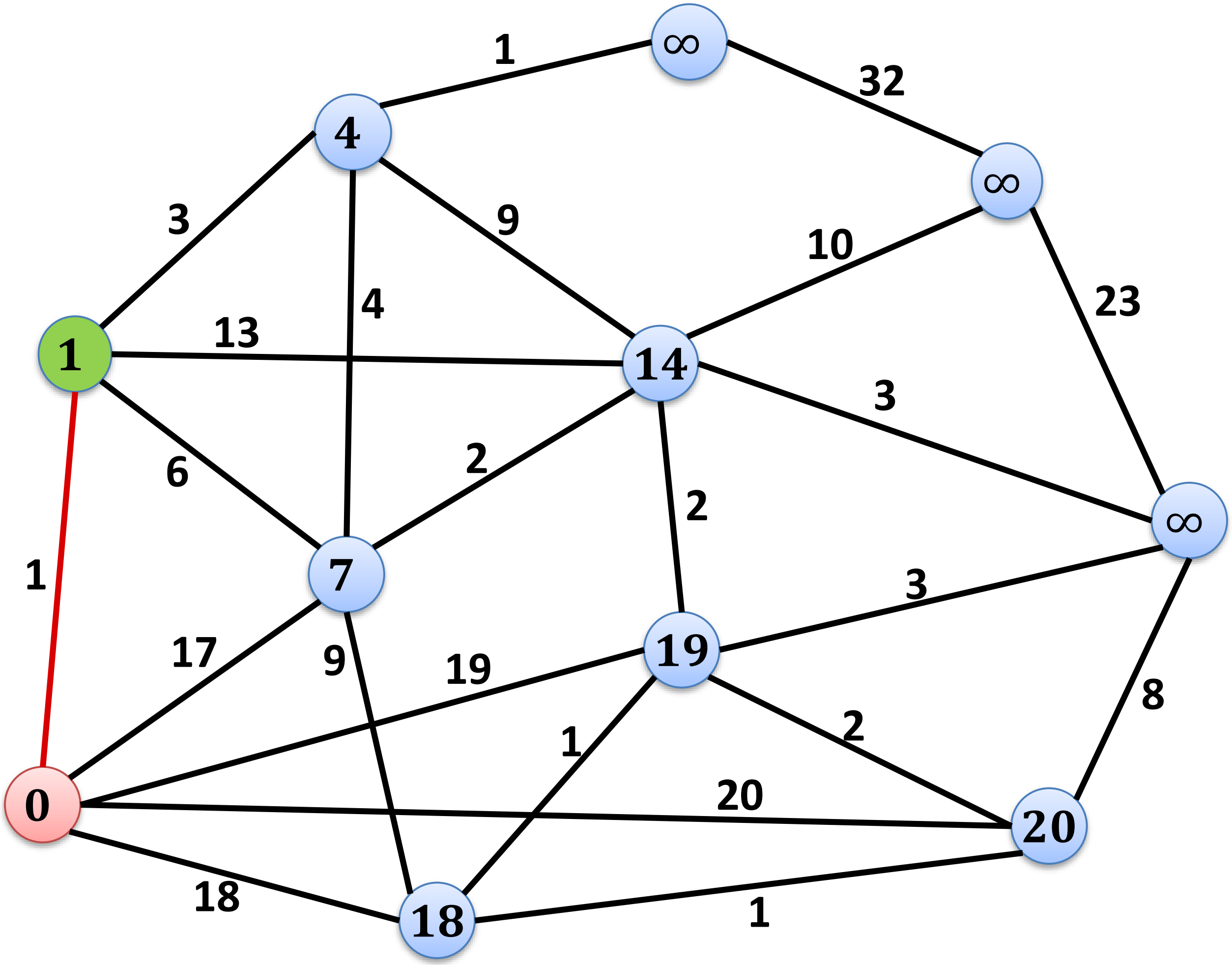

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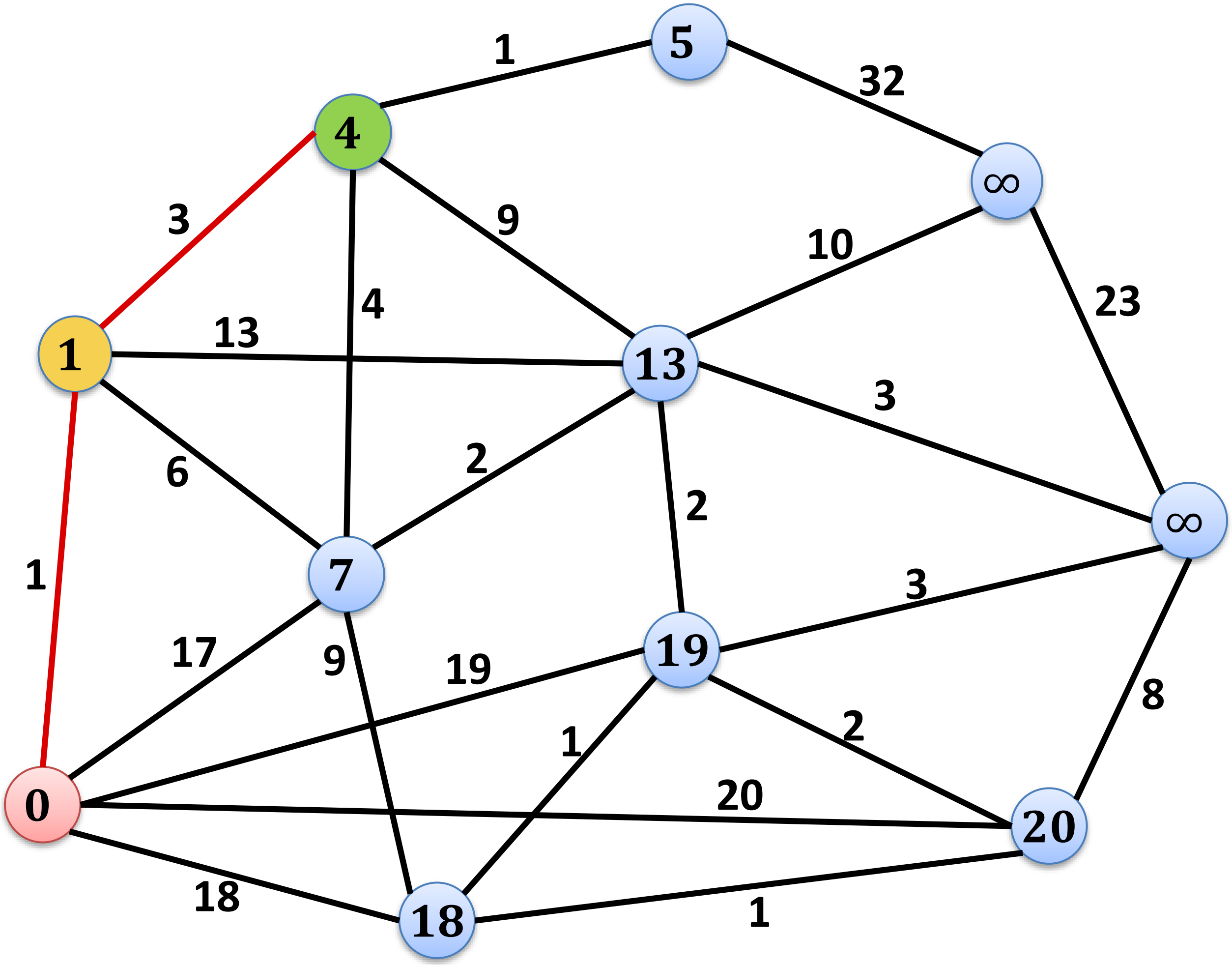

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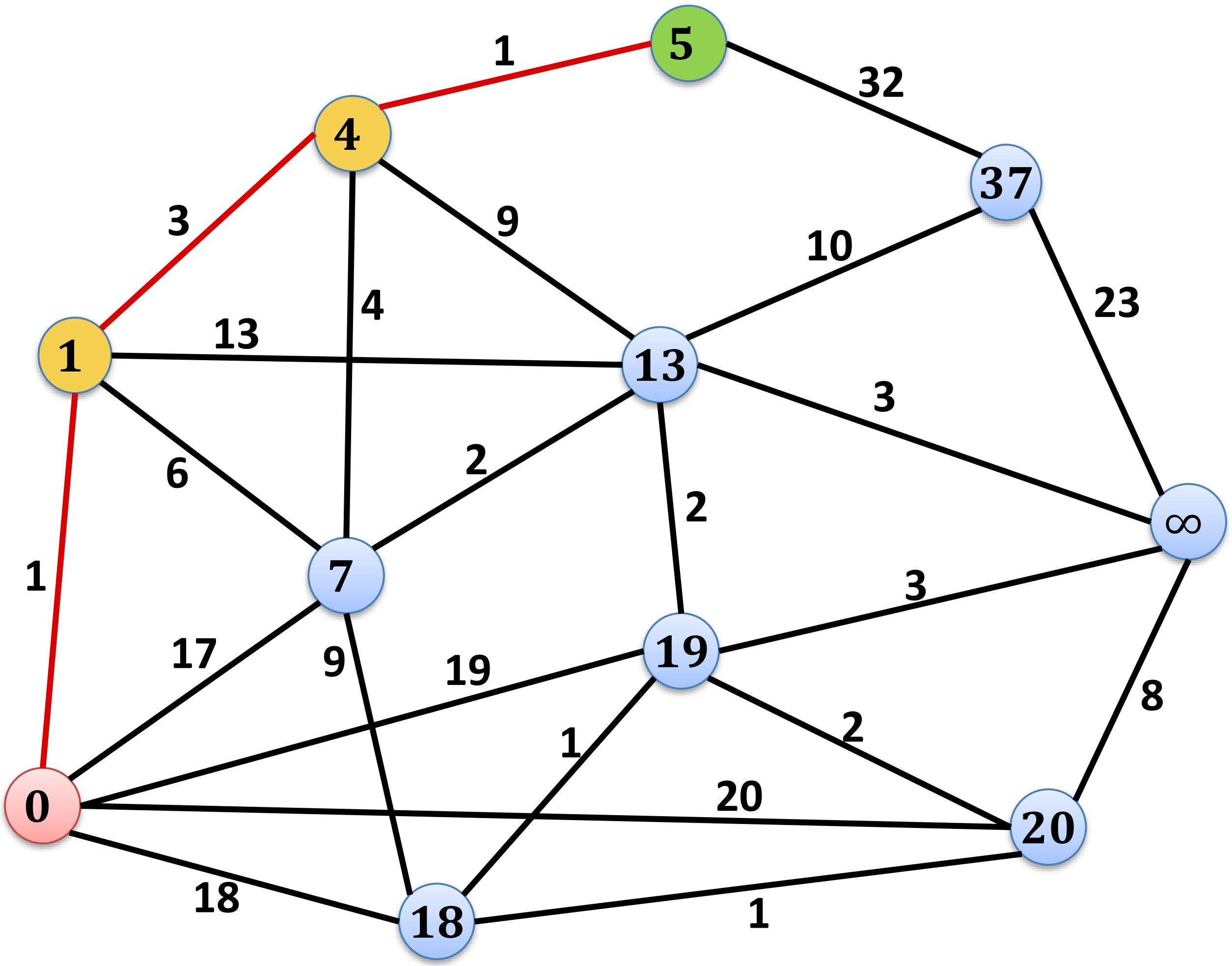

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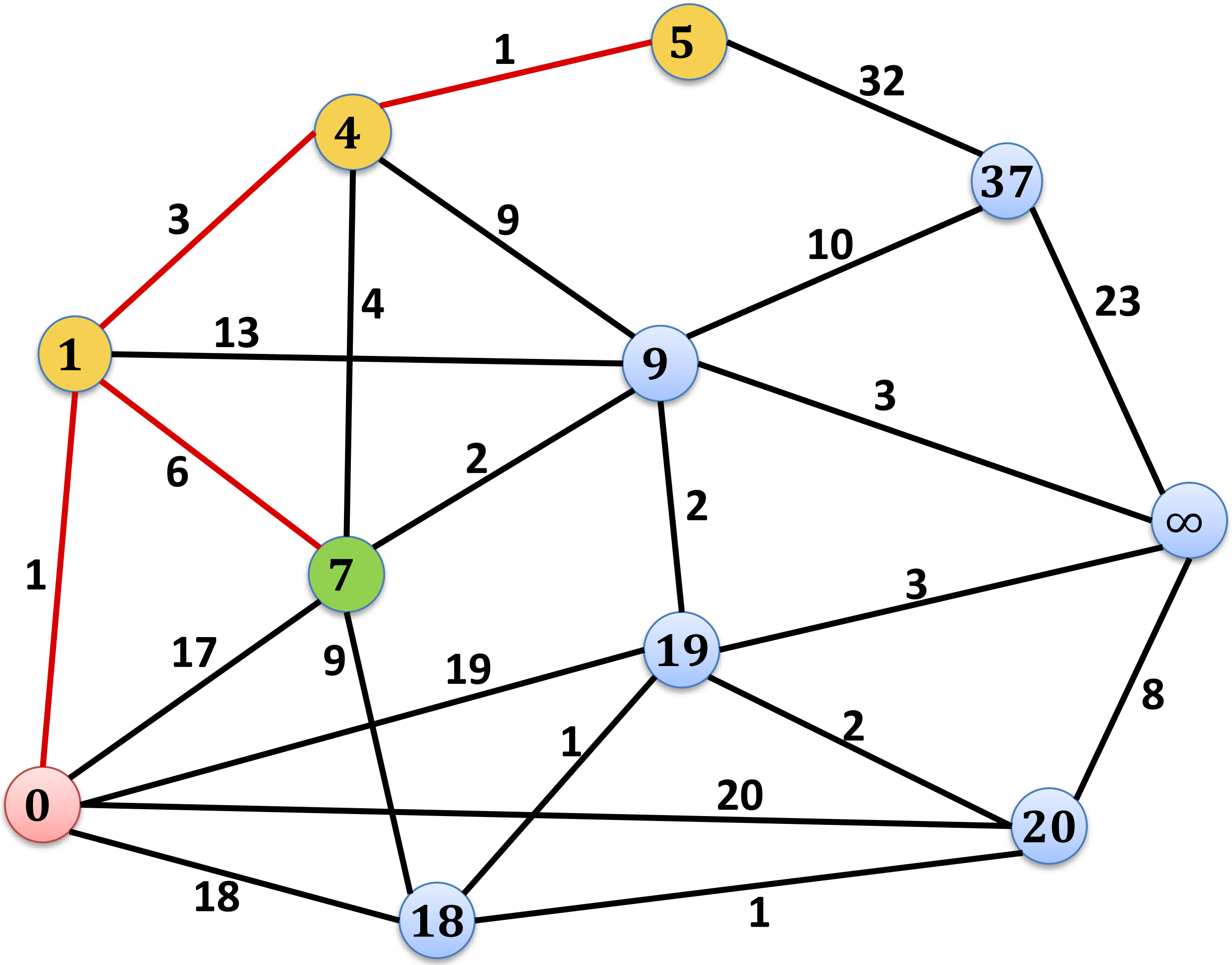

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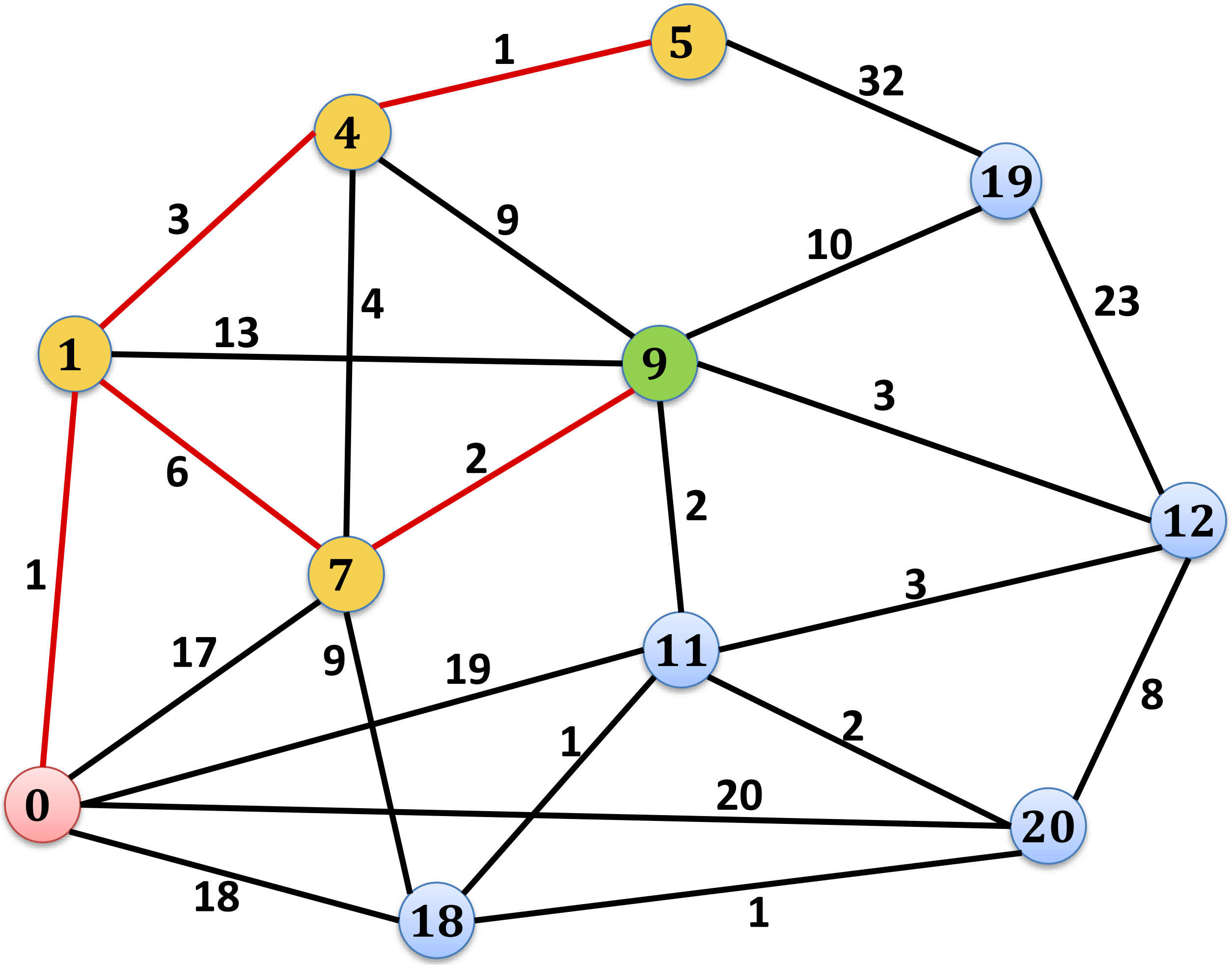

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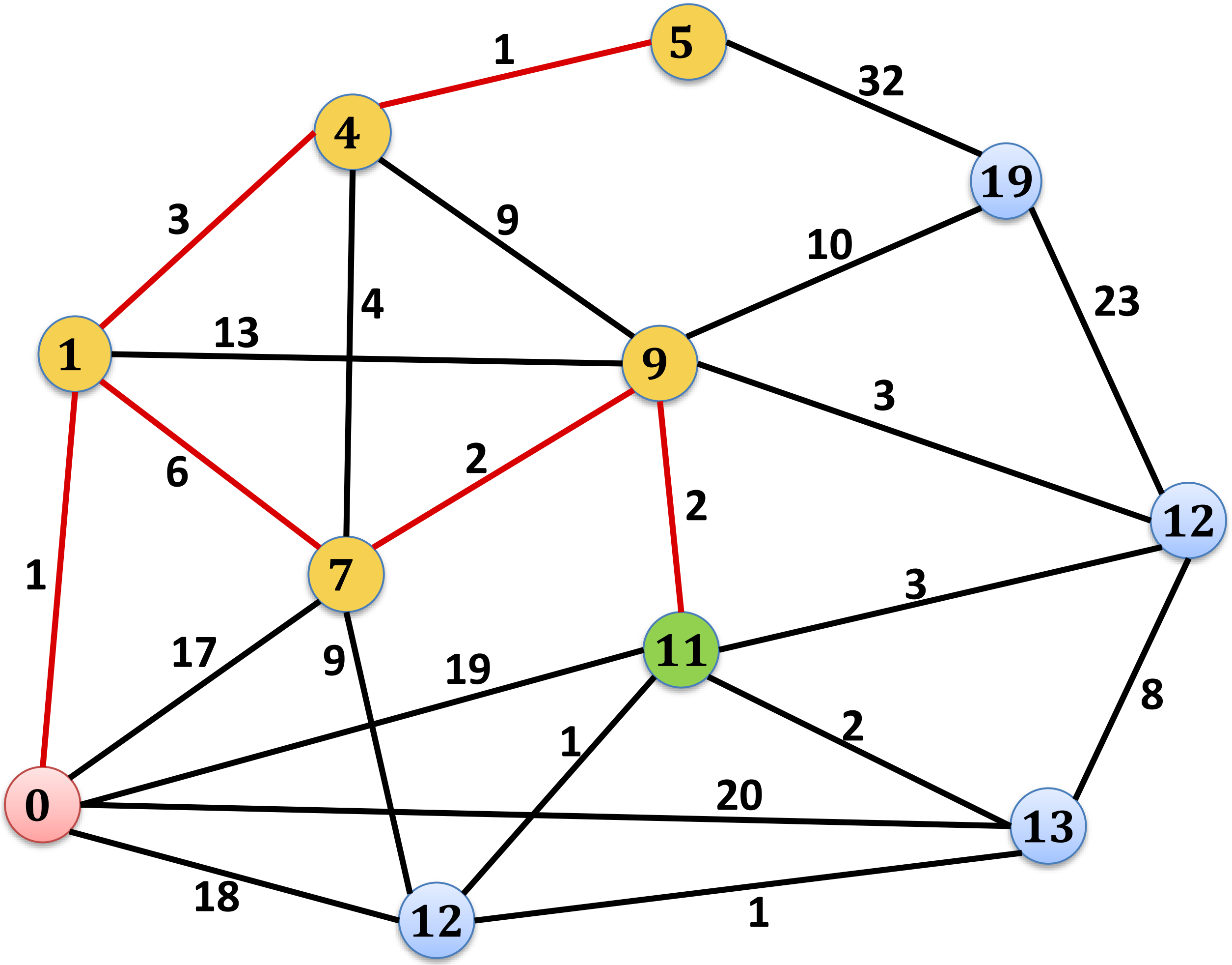


# Implementation of Dijkstra's Algorithm

**Dijkstra's Algorithm:**

```
1. Initialize $d(s,s) = 0$ and $d(s,v) = \infty$ for all $v \neq s$
2. All nodes $v \neq s$ are unmarked

3. Get unmarked node $u$ which minimizes $d(s,u)$:

4.     For all $e = \{u,v\} \in E$, $d(s,v) = \min\{d(s,v), d(s,u) + w(e)\}$

5.     mark node $u$

6. Until all nodes are marked
```

# Priority Queue / Heap

- Stores (*key,data*) pairs (like dictionary)
- But, different set of operations:

- **Initialize-Heap**: creates new empty heap
- **Is-Empty**: returns true if heap is empty
- **Insert**(*key,data*): inserts (*key,data*)-pair, returns pointer to entry
- **Get-Min**: returns (*key,data*)-pair with minimum *key*
- **Delete-Min**: deletes minimum (*key,data*)-pair
  - has to be consistent with get-min operation
- **Decrease-Key**(*entry,newkey*): decreases *key* of *entry* to *newkey*
- **Merge**: merges two heaps into one

# Implementation of Dijkstra's Algorithm

**Store nodes in a priority queue, use $d(s,v)$ as keys:**

```
1. Initialize $d(s,s) = 0$ and $d(s,v) = \infty$ for all $v \neq s$
2. All nodes $v \neq s$ are unmarked

3. Get unmarked node $u$ which minimizes $d(s,u)$:

4.     mark node $u$

5.     For all $e = \{u,v\} \in E$, $d(s,v) = \min\{d(s,v), d(s,u) + w(e)\}$

6. Until all nodes are marked
```

# Analysis

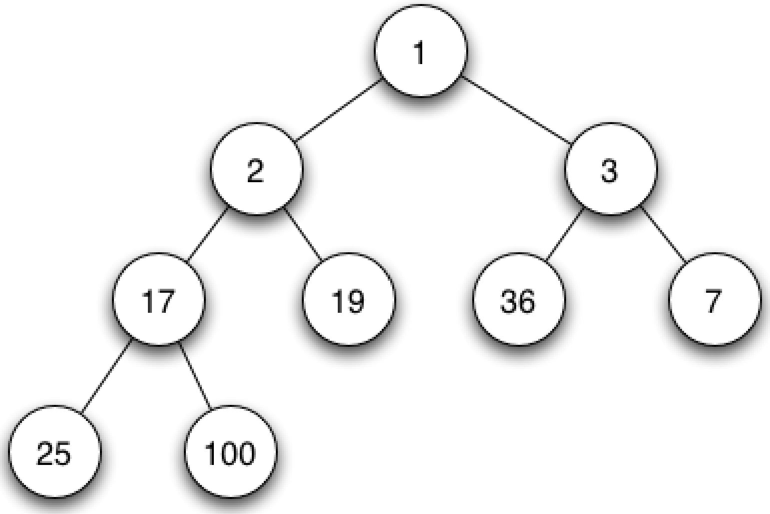
Number of priority queue operations for Dijkstra:

- **Initialize-Heap**: $\mathbf{1}$
- **Is-Empty**: $|\boldsymbol{V}|$
- **Insert**: $|\boldsymbol{V}|$
- **Get-Min**: $|\boldsymbol{V}|$
- **Delete-Min**: $|\boldsymbol{V}|$
- **Decrease-Key**: $|\boldsymbol{E}|$
- **Merge**: $\mathbf{0}$

# Priority Queue Implementation

Implementation as min-heap:

→ complete binary tree,
e.g., stored in an array



# Priority Queue Implementation

Implementation as min-heap:

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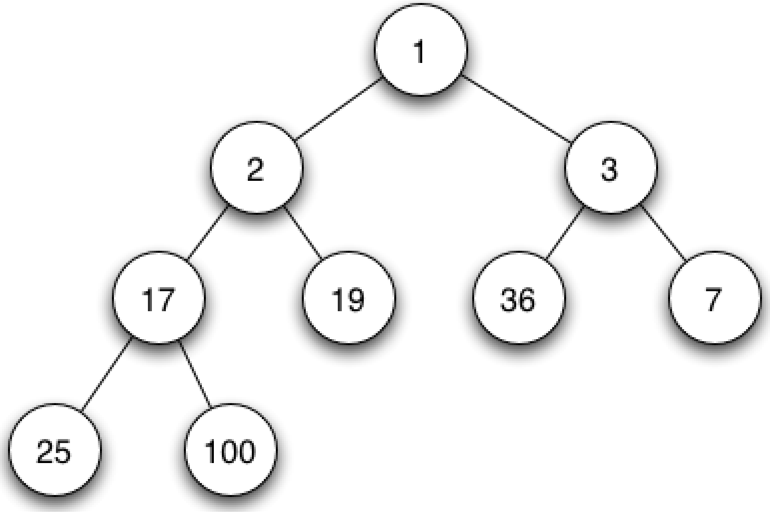


- **Initialize-Heap**: $\boldsymbol{O(1)}$
- **Is-Empty**: $\boldsymbol{O(1)}$
- **Insert**: $\boldsymbol{O(\log n)}$
- **Get-Min**: $\boldsymbol{O(1)}$
- **Delete-Min**: $\boldsymbol{O(\log n)}$
- **Decrease-Key**: $\boldsymbol{O(\log n)}$
- **Merge** (heaps of size $m$ and $n$, $m \leq n$): $\boldsymbol{O(m \log n)}$

# Can We Do Better?

- Cost of **Dijkstra** with **complete binary min-heap** implementation:

$$O(|E| \log|V|)$$

- **Binary heap:**
  insert, delete-min, and decrease-key cost $O(\log n)$
  merging two heaps is expensive

- One of the operations insert or delete-min must cost $\Omega(\log n)$:
  - Heap-Sort:
    Insert $n$ elements into heap, then take out the minimum $n$ times
  - (Comparison-based) sorting costs at least $\Omega(n \log n)$.

- But maybe we can improve merge, decrease-key, and one of the other two operations?

# Fibonacci Heaps

**Structure:**

A Fibonacci heap $H$ consists of a collection of trees satisfying the **min-heap** property.

**Min-Heap Property:**

Key of a node $v \leq$ keys of all nodes in any sub-tree of $v$

# Fibonacci Heaps

**Structure:**

A Fibonacci heap $H$ consists of a collection of trees satisfying the min-heap property.

**Variables:**

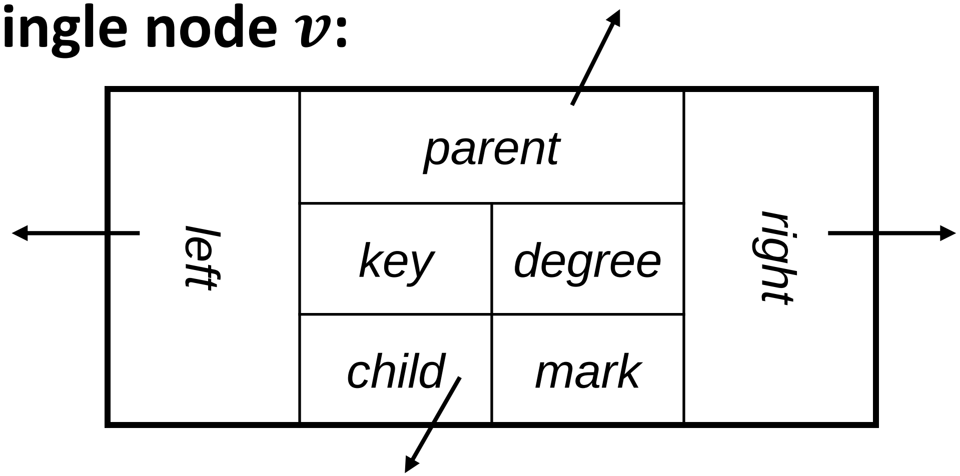
- $H.min$: root of the tree containing the (a) minimum key
- $H.rootlist$: circular, doubly linked, unordered list containing the roots of all trees
- $H.size$: number of nodes currently in $H$

**Lazy Merging:**

- To reduce the number of trees, sometimes, trees need to be merged
- Lazy merging: Do not merge as long as possible...

# Trees in Fibonacci Heaps

**Structure of a single node $v$:**



- $v.child$: points to circular, doubly linked and unordered list of the children of $v$
- $v.left, v.right$: pointers to siblings (in doubly linked list)
- $v.mark$: will be used later…

**Advantages of circular, doubly linked lists:**

- Deleting an element takes constant time
- Concatenating two lists takes constant time

# Example

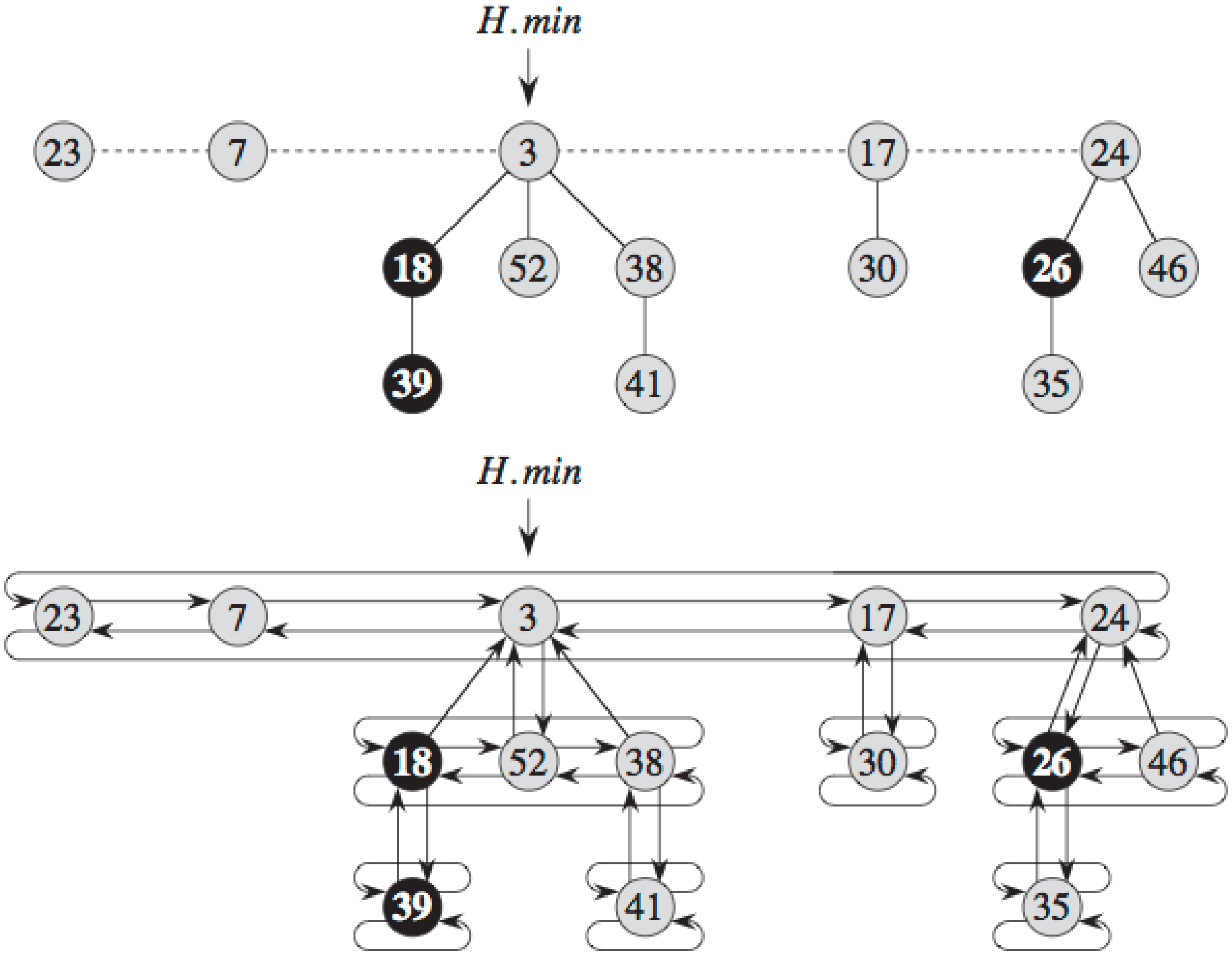


Figure: Cormen et al., Introduction to Algorithms

# Simple (Lazy) Operations

**Initialize-Heap** $H$:

- $H.rootlist \coloneqq H.min \coloneqq null$

**Merge** heaps $H$ and $H'$:

- concatenate root lists
- update $H.min$

**Insert** element $e$ into $H$:

- create new one-node tree containing $e$ → $\mathrm{H}'$
  - mark of root node is set to $\mathbf{false}$
- merge heaps $H$ and $H'$

**Get minimum** element of $H$:

- return $H.min$

# Operation Delete-Min

Delete the node with minimum key from $H$ and return its element:

```
1.  $m := H.min$;
2.  if $H.size > 0$ then
3.      remove $H.min$ from $H.rootlist$;
4.      add $H.min.child$ (list) to $H.rootlist$
5.  $\boldsymbol{H.Consolidate()}$;

    // Repeatedly merge nodes with equal degree in the root list
    // until degrees of nodes in the root list are distinct.
    // Determine the element with minimum key

6.  return $m$
```

# Rank and Maximum Degree

**Ranks of nodes, trees, heap:**

Node $v$:

- $rank(v)$: degree of $v$ (number of children of $v$)

Tree $T$:

- $rank(T)$: rank (degree) of root node of $T$

Heap $H$:

- $rank(H)$: maximum degree (#children) of any node in $H$

**Assumption** ($n$: number of nodes in $H$):

$$rank(H) \leq D(n)$$

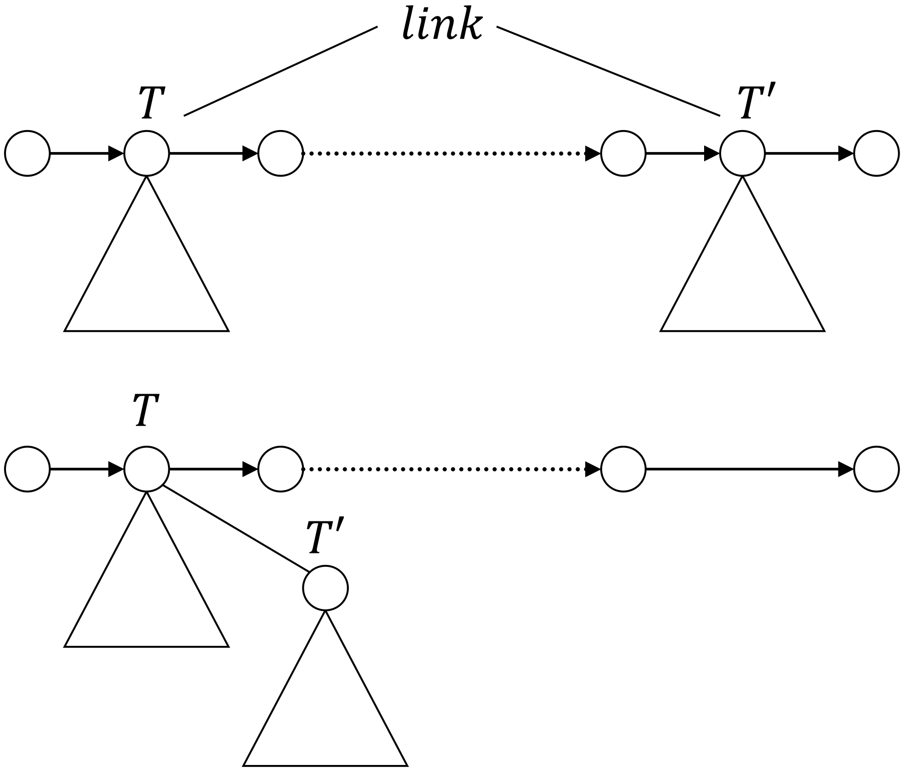
  - for a known function $D(n)$

# Merging Two Trees

**Given:** Heap-ordered trees $T, T'$ with $rank(T) = rank(T')$

- Assume: min-key of $T <$ min-key of $T'$

**Operation $\boldsymbol{link(T, T')}$:**

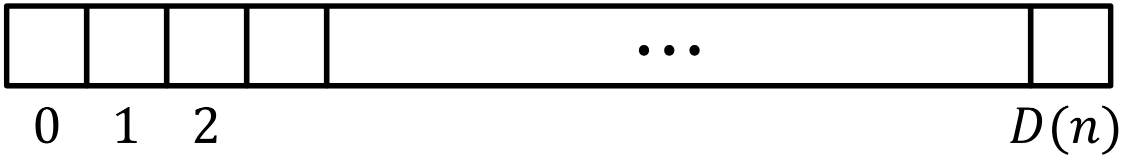


- Removes tree $T'$ from root list and adds $T'$ to child list of $T$

- $rank(T) \coloneqq rank(T) + 1$
- $(T'.mark = \textbf{false})$

# Consolidation of Root List

Array $A$ pointing to find roots with the same rank:



**Consolidate:**

**Time:** $O(|H.rootlist| + D(n))$

```
for $i := 0$ to $D(n)$ do $A[i] :=$ null;
while $H.rootlist \neq$ null do
    $T :=$ “delete and return first element of $H.rootlist$”
    while $A[rank(T)] \neq$ null do
        $T' := A[rank(T)]$;
        $A[rank(T)] := null$;
        $T := link(T, T')$
    $A[rank(T)] := T$
Create new $H.rootlist$ and $H.min$
```