Chapter 6
Graph Algorithms

Algorithm Theory
WS 2019/20

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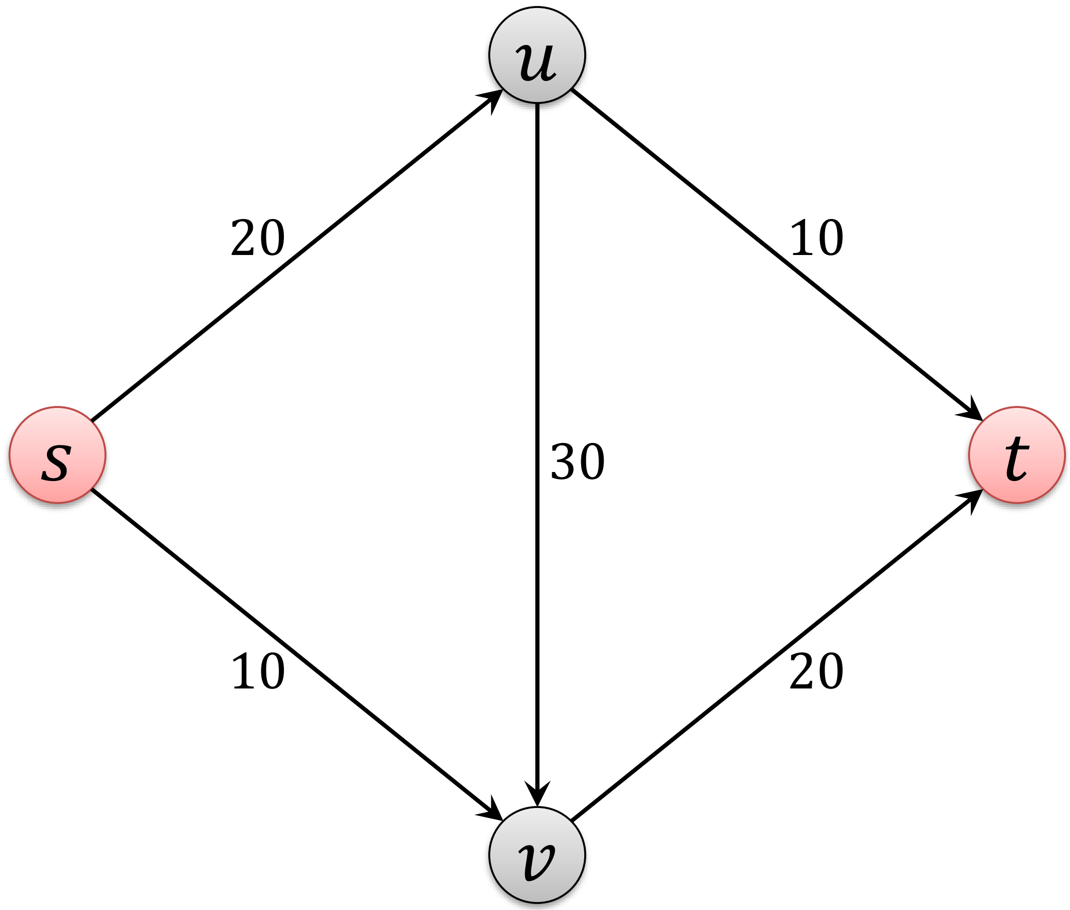
Example: Flow Network

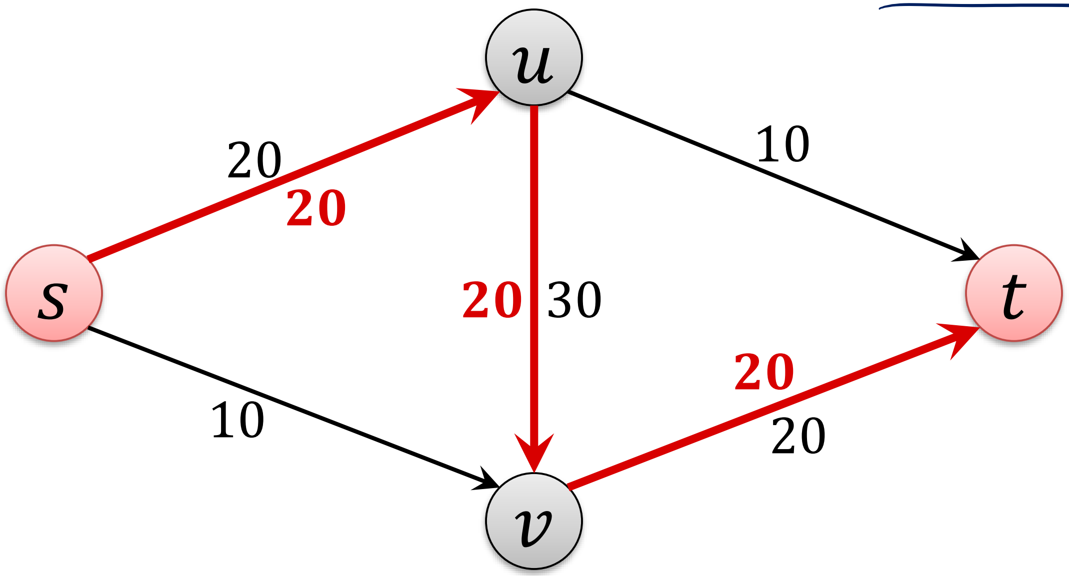


Residual Graph

Given a flow network $G = (V, E)$ with capacities c_e (for $e \in E$)

For a flow f on G, define directed graph $G_f = (V_f, E_f)$ as follows:

- Node set $V_f = V$
- For each edge $e = (u, v)$ in E, there are two edges in E_f:
  - forward edge $e = (u, v)$ with residual capacity $c_e - f(e)$
  - backward edge $e' = (v, u)$ with residual capacity $f(e)$



Augmenting Path

Definition:
An augmenting path P is a (simple) s-t-path on the residual graph G_f on which each edge has residual capacity > 0.

$\text{bottleneck}(P, f)$: minimum residual capacity on any edge of the augmenting path P

Augment flow f to get flow f':

- For every forward edge (u, v) on P:

$$f'\big((u, v)\big) := f\big((u, v)\big) + \textbf{bottleneck}(P, f)$$

- For every backward edge (u, v) on P:

$$f'\big((v, u)\big) := f\big((v, u)\big) - \textbf{bottleneck}(P, f)$$

Ford-Fulkerson Algorithm

- Improve flow using an augmenting path as long as possible:

1. Initially, $f(e) = 0$ for all edges $e \in E$, $G_f = G$
2. **while** there is an augmenting s-t-path P in G_f **do**
3. Let P be an augmenting s-t-path in G_f;
4. $f' \coloneqq \mathrm{augment}(f, P)$;
5. update f to be f';
6. update the residual graph G_f
7. **end**;

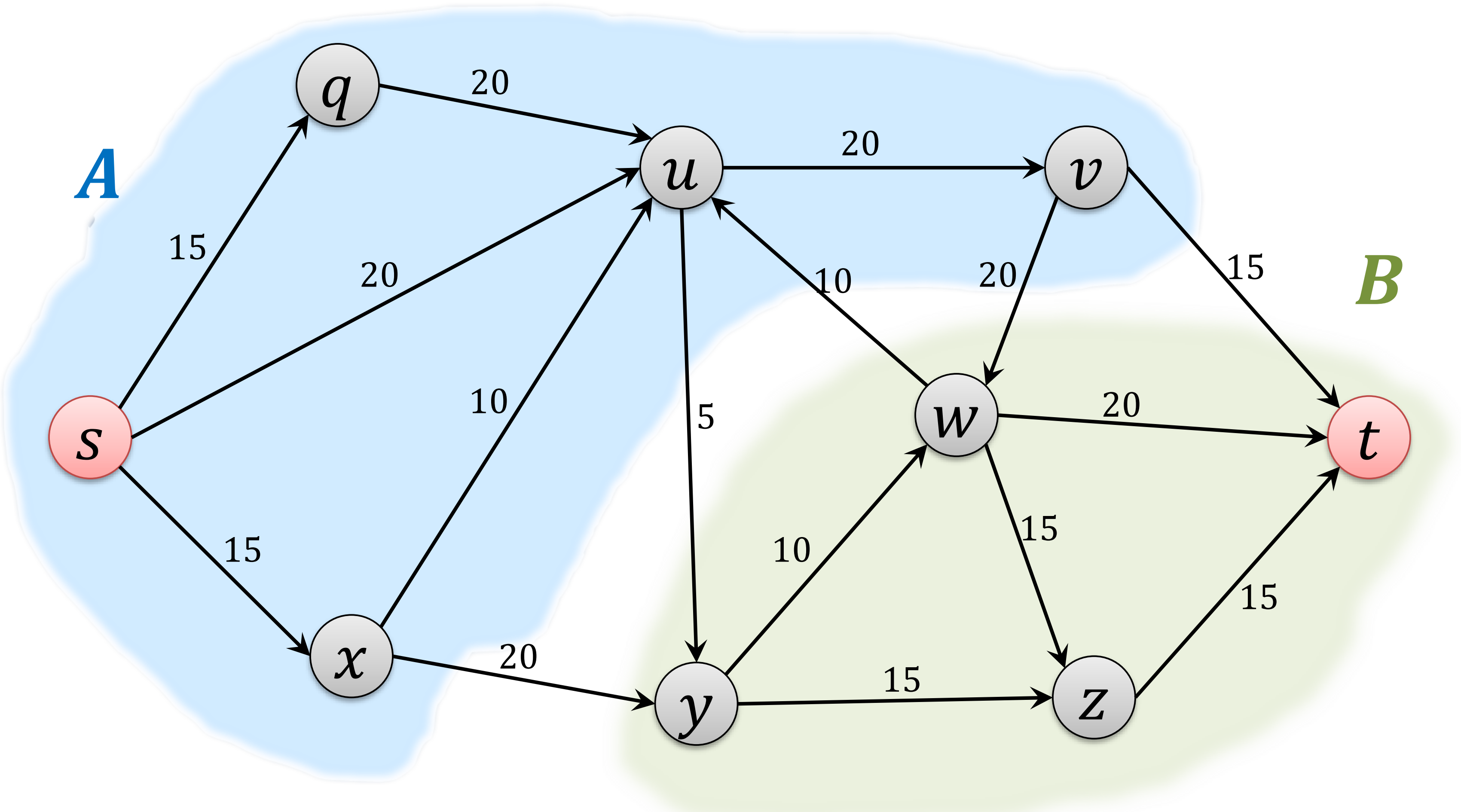
Ford-Fulkerson Running Time

Theorem: If all edge capacities are integers, the Ford-Fulkerson algorithm can be implemented to run in $O(mC)$ time.

Proof:

s-t Cuts

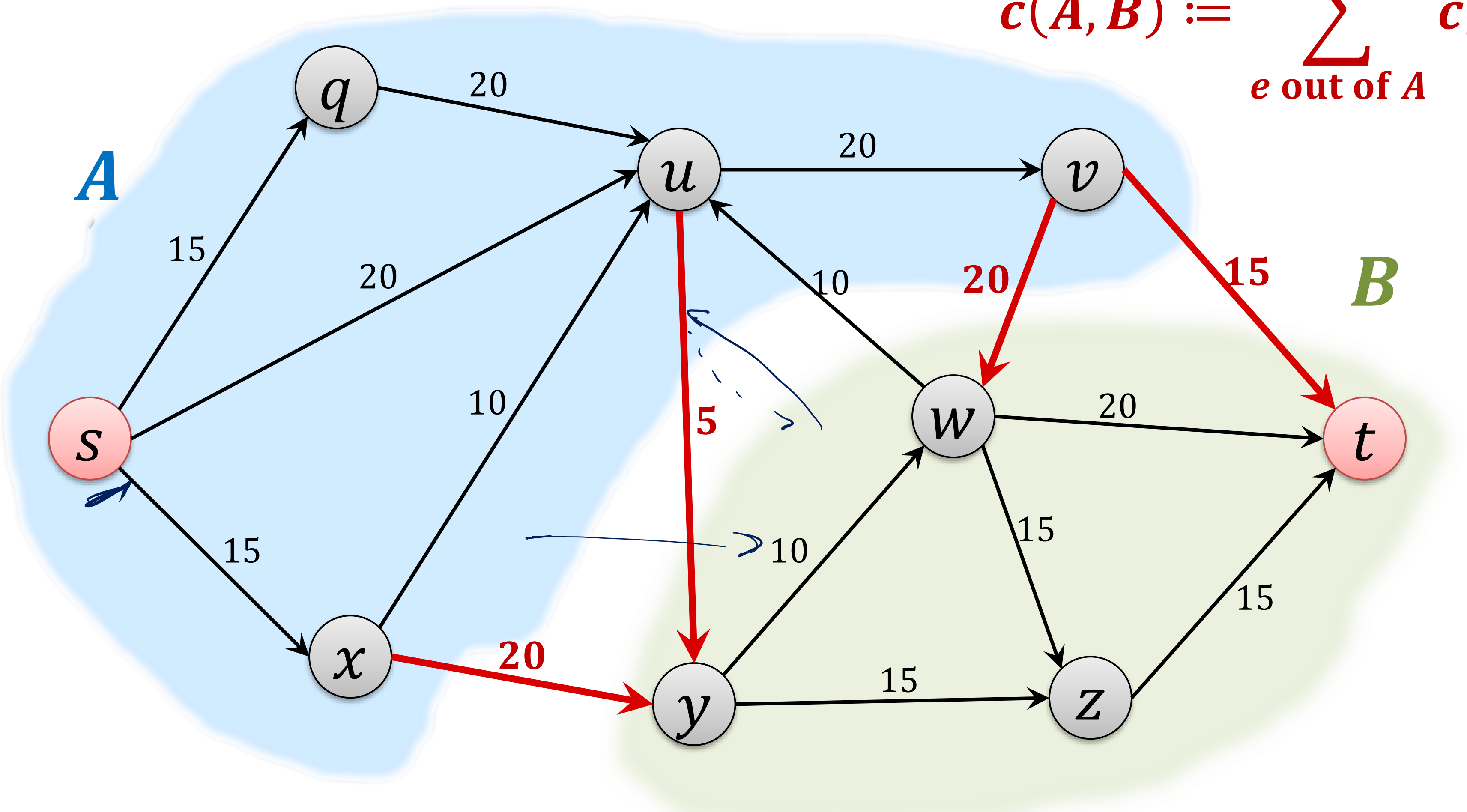
Definition:
An s-t cut is a partition (A, B) of the vertex set such that $s \in A$ and $t \in B$



Cut Capacity

Definition:
The capacity $c(A, B)$ of an s-t-cut (A, B) is defined as

$$c(A,B) := \sum_{e \text{ out of } A} c_e .$$



Max-Flow Min-Cut Theorem

Theorem: (Max-Flow Min-Cut Theorem)

In every flow network, the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.

Proof:

Strongly Polynomial Algorithm

- Time of regular Ford-Fulkerson algorithm with integer capacities:

$$O(mC)$$

- Time of algorithm with scaling parameter:

$$O(m^2 \log C)$$

- $O(\log C)$ is polynomial in the size of the input, but not in n
- Can we get an algorithm that runs in time polynomial in n?
- Always picking a shortest augmenting path leads to running time

$$O(m^2 n)$$

  - also works for arbitrary real-valued weights

Other Algorithms

- There are many other algorithms to solve the maximum flow problem, for example:

- **Preflow-push algorithm:**
  - Maintains a preflow (∀ nodes: inflow ≥ outflow)
  - Alg. guarantees: As soon as we have a flow, it is optimal
  - Detailed discussion in 2012/13 lecture
  - Running time of basic algorithm: $O(m \cdot n^2)$
  - Doing steps in the "right" order: $O(n^3)$

- **Current best known complexity:** $\boldsymbol{O(m \cdot n)}$
  - For graphs with $m \geq n^{1+\epsilon}$ (for every constant $\epsilon > 0$) [King,Rao,Tarjan 1992/1994]
  - For sparse graphs with $m \leq n^{16/15-\delta}$ [Orlin, 2013]

Maximum Flow Applications

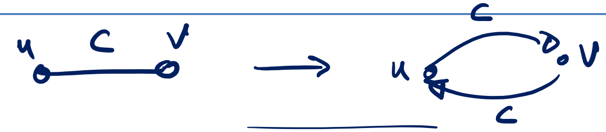
- Maximum flow has many applications

- Reducing a problem to a max flow problem can even be seen as an important algorithmic technique

- Examples:
  - related network flow problems
  - computation of small cuts
  - computation of matchings
  - computing disjoint paths
  - scheduling problems
  - assignment problems with some side constraints
  - …

Undirected Edges and Vertex Capacities

Undirected Edges:



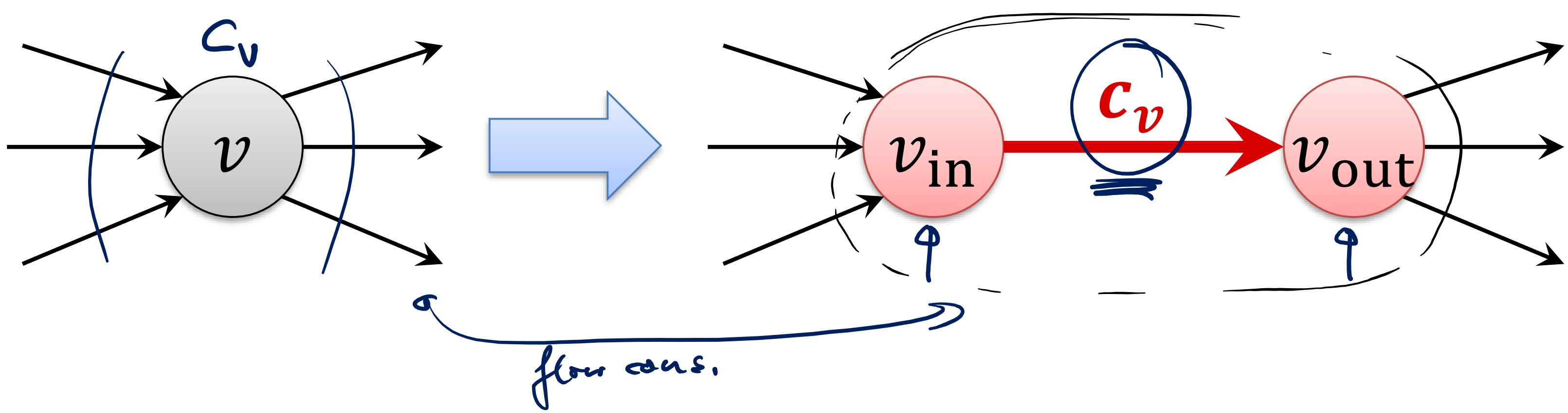
- Undirected edge $\{u, v\}$: add edges (u, v) and (v, u) to network

Vertex Capacities:

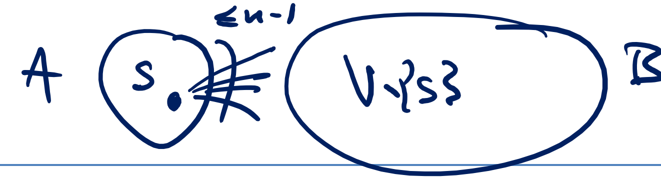
- Not only edges, but also (or only) nodes have capacities
- Capacity c_v of node $v \notin \{s, t\}$:

$$f^{\text{in}}(v) = f^{\text{out}}(v) \leq c_v$$

- Replace node v by edge $e_v = \{v_{\text{in}}, v_{\text{out}}\}$:



Minimum s-t Cut



Given: undirected graph $G = (V, E)$, nodes $s, t \in V$

s-t cut: Partition (A, B) of V such that $s \in A$, $t \in B$

Size of cut (A, B): number of edges crossing the cut

Size of cut = # edges crossing the cut

running time:

$O(m \cdot n)$

$C \leq n-1$

Objective: find s-t cut of minimum size

create flow network: 1) make edges directed

2) edge cap. = 1

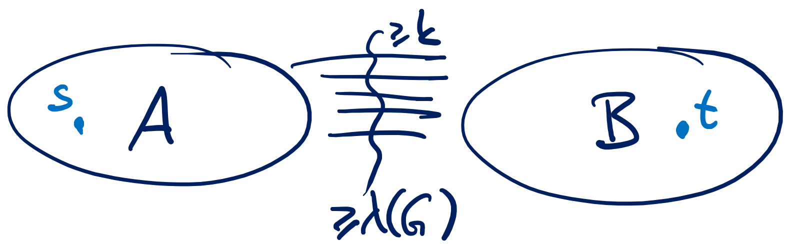
Size of cut in G = cap. cut in flow network

Edge Connectivity

Definition: A graph $G = (V, E)$ is k-edge connected for an integer $k \geq 1$ if the graph $G_X = (V, E \setminus X)$ is connected for every edge set

$$X \subseteq E, |X| \leq k - 1.$$

need to remove at least k edges to disconnect G.



edge connectivity λ(G): max k s.t. G is k-edge connected

Goal: Compute edge connectivity $\lambda(G)$ of G
(and edge set X of size $\lambda(G)$ that divides G into ≥ 2 parts)

- minimum set X is a minimum s-t cut for some $s, t \in V$
  - Actually for all s, t in different components of $G_X = (V, E \setminus X)$
- Possible algorithm: fix s and find min s-t cut for all $t \neq s$

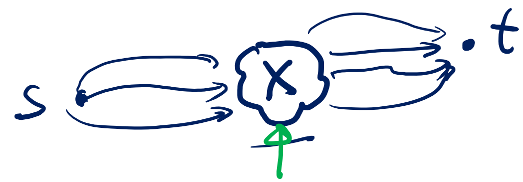
time: $O(m \cdot n^2)$

Minimum s-t Vertex-Cut

Given: undirected graph $G = (V, E)$, nodes $s, t \in V$

***s*-*t* vertex cut:** Set $X \subset V$ such that $s, t \notin X$ and s and t are in different components of the sub-graph $G[V \setminus X]$ induced by $V \setminus X$

Size of vertex cut: $|X|$



Objective: find s-t vertex-cut of minimum size

- Replace undirected edge $\{u, v\}$ by (u, v) and (v, u)
- Compute max s-t flow for edge capacities ∞ and node capacities

$$c_v = 1 \text{ for } v \neq s, t$$

- Replace each node v by v_{in} and v_{out}:

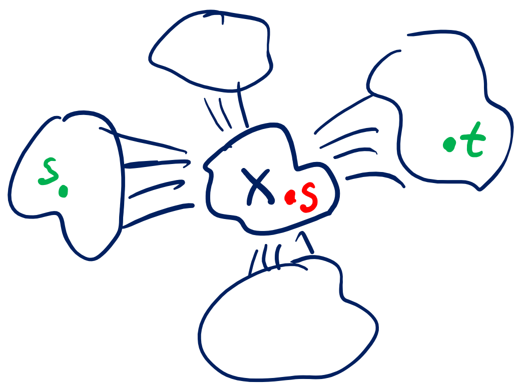
- Min edge cut corresponds to min vertex cut in G

Vertex Connectivity

Definition: A graph $G = (V, E)$ is k-vertex connected for an integer $k \geq 1$ if the sub-graph $G[V \setminus X]$ induced by $V \setminus X$ is connected for every edge set

$$X \subseteq V, |X| \leq k - 1.$$

need to remove at least k nodes to disconnect G



vertex connectivity $\kappa(G)$:
max k s.t.
G is k-vertex conn.

Goal: Compute vertex connectivity $\kappa(G)$ of G

(and node set X of size $\kappa(G)$ that divides G into ≥ 2 parts)

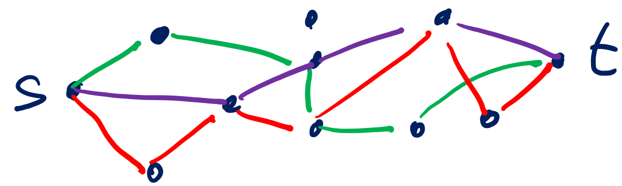
- Compute minimum s-t vertex cut for all s and all $t \neq s$

$O(m \cdot n^3)$

Edge-Disjoint Paths

Given: Graph $G = (V, E)$ with nodes $s, t \in V$

Goal: Find as many edge-disjoint s-t paths as possible



Solution:

- Find max s-t flow in G with edge capacities $c_e = 1$ for all $e \in E$

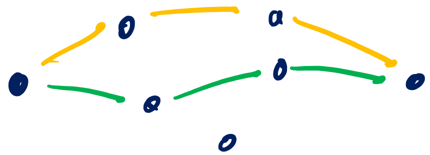
Flow f induces $|f|$ edge-disjoint paths:

- Integral capacities → can compute integral max flow f
- Get $|f|$ edge-disjoint paths by greedily picking them
- Correctness follows from flow conservation $f^{\text{in}}(v) = f^{\text{out}}(v)$

Vertex-Disjoint Paths

Given: Graph $G = (V, E)$ with nodes $s, t \in V$

Goal: Find as many internally vertex-disjoint s-t paths as possible



Solution:

- Find max s-t flow in G with node capacities $c_v = 1$ for all $v \in V$

Flow f induces $|f|$ vertex-disjoint paths:

- Integral capacities → can compute integral max flow f
- Get $|f|$ vertex-disjoint paths by greedily picking them
- Correctness follows from flow conservation $f^{\text{in}}(v) = f^{\text{out}}(v)$

Menger's Theorem

Theorem: (edge version)
For every graph $G = (V, E)$ with nodes $s, t \in V$, the size of the minimum s-t (edge) cut equals the maximum number of pairwise edge-disjoint paths from s to t.

Theorem: (node version)
For every graph $G = (V, E)$ with nodes $s, t \in V$, the size of the minimum s-t vertex cut equals the maximum number of pairwise internally vertex-disjoint paths from s to t

- Both versions can be seen as a special case of the max flow min cut theorem

Baseball Elimination

Team i	Wins w_i	Losses ℓ_i	To Play r_i	Against = r_{ij}				
				NY	Balt.	T. Bay	Tor.	Bost.
New York	81	69	12	-	2	5	2	3
Baltimore	79	77	6	2	-	2	1	1
Tampa Bay	79	74	9	5	2	-	1	1
Toronto	76	80	6	2	1	1	-	2
Boston	71	84	7	3	1	1	2	-

- Only wins/losses possible (no ties), winner: team with most wins
- Which teams can still win (as least as many wins as top team)?
- Boston is eliminated (cannot win):
  - Boston can get at most 78 wins, New York already has 81 wins
- If for some i, j: $w_i + r_i < w_j$ → team i is eliminated
- Sufficient condition, but not a necessary one!

Baseball Elimination

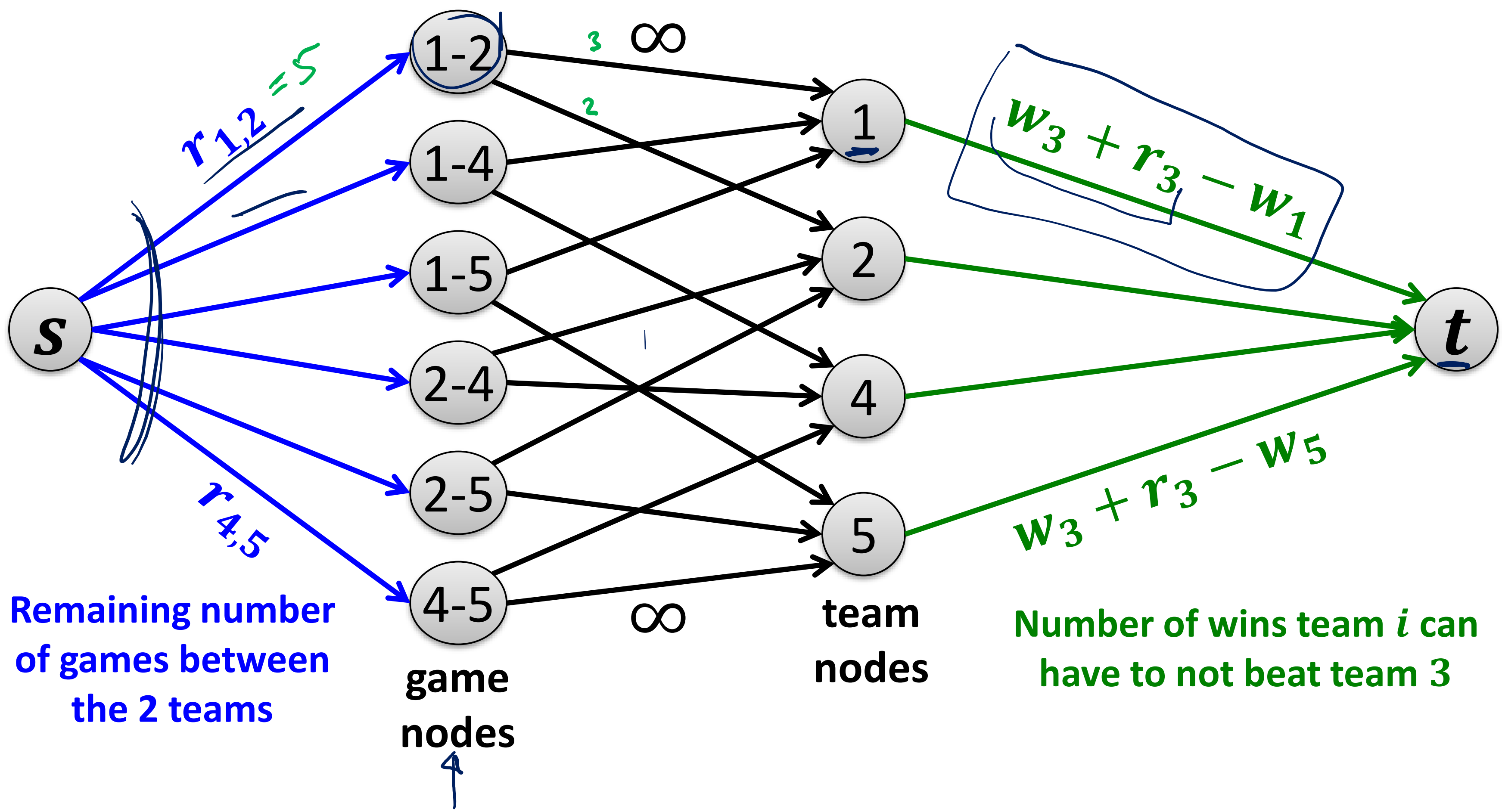
Team i	Wins w_i	Losses ℓ_i	To Play r_i	Against = r_{ij}				
				NY	Balt.	T. Bay	Tor.	Bost.
New York	81	69	12	-	2	5	2	3
Baltimore	79	77	6	2	-	2	1	1
Tampa Bay	79	74	9	5	2	-	1	1
Toronto	76	80	6	2	1	1	-	2
Boston	71	84	7	3	1	1	2	-

- Can Toronto still finish first?
- Toronto can get $82 > 81$ wins, but:
  NY and Tampa have to play 5 more times against each other
  → if NY wins two, it gets 83 wins, otherwise, Tampa has 83 wins
- Hence: Toronto cannot finish first
- How about the others? How can we solve this in general?

Max Flow Formulation

w_i : # wins so far
r_i : # remaining games

- Can team 3 finish with most wins?



- Team 3 can finish first iff all source-game edges are saturated

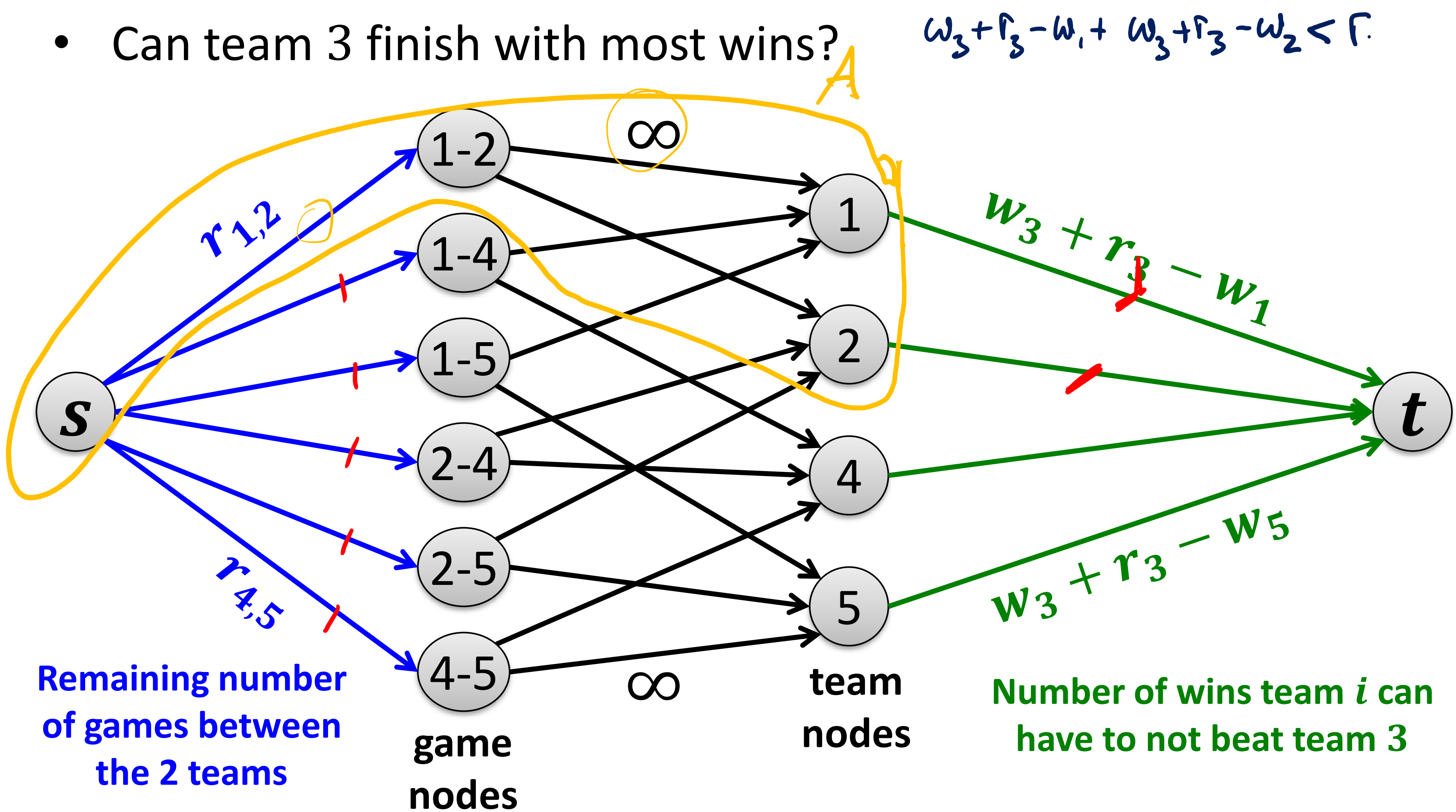
Reason for Elimination

AL East: Aug 30, 1996

Team i	Wins w_i	Losses ℓ_i	To Play r_i	Against = r_{ij}				
				NY	Balt.	Bost.	Tor.	Detr.
New York	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	0
Detroit	49	86	27	3	4	0	0	-

- Detroit could finish with $49 + 27 = 76$ wins
- Consider $R = \{\text{NY}, \text{Bal}, \text{Bos}, \text{Tor}\}$
  - Have together already won $w(R) = 278$ games
  - Must together win at least $r(R) = 27$ more games
- On average, teams in R win $\frac{278+27}{4} = 76.25$ games

- Can team 3 finish with most wins?



- Team 3 cannot finish first ⟺ min cut of size < "all blue edges"

Reason for Elimination

Certificate of elimination:

$$R \subseteq X, \qquad w(R) := \underbrace{\sum_{i \in R} w_i}_{\text{\#wins of nodes in } R}, \qquad r(R) := \underbrace{\sum_{i,j \in R} r_{i,j}}_{\text{\#remaining games among nodes in } R}$$

Team $x \in X$ is eliminated by R if

$$\frac{w(R) + r(R)}{|R|} > w_x + r_x.$$

Reason for Elimination

Theorem: Team x is eliminated if and only if there exists a subset $R \subseteq X$ of the teams X such that x is eliminated by R.

Proof Idea:

- Minimum cut gives a certificate…
- If x is eliminated, max flow solution does not saturate all outgoing edges of the source.
- Team nodes of unsaturated source-game edges are saturated
- Source side of min cut contains all teams of saturated team-dest. edges of unsaturated source-game edges
- Set of team nodes in source-side of min cut give a certificate R

Circulations with Demands

Given: Directed network with positive edge capacities

Sources & Sinks: Instead of one source and one destination, several sources that generate flow and several sinks that absorb flow.

Supply & Demand: sources have supply values, sinks demand values

Goal: Compute a flow such that source supplies and sink demands are exactly satisfied

- The circulation problem is a feasibility rather than a maximization problem

Circulations with Demands: Formally

Given: Directed network $G = (V, E)$ with

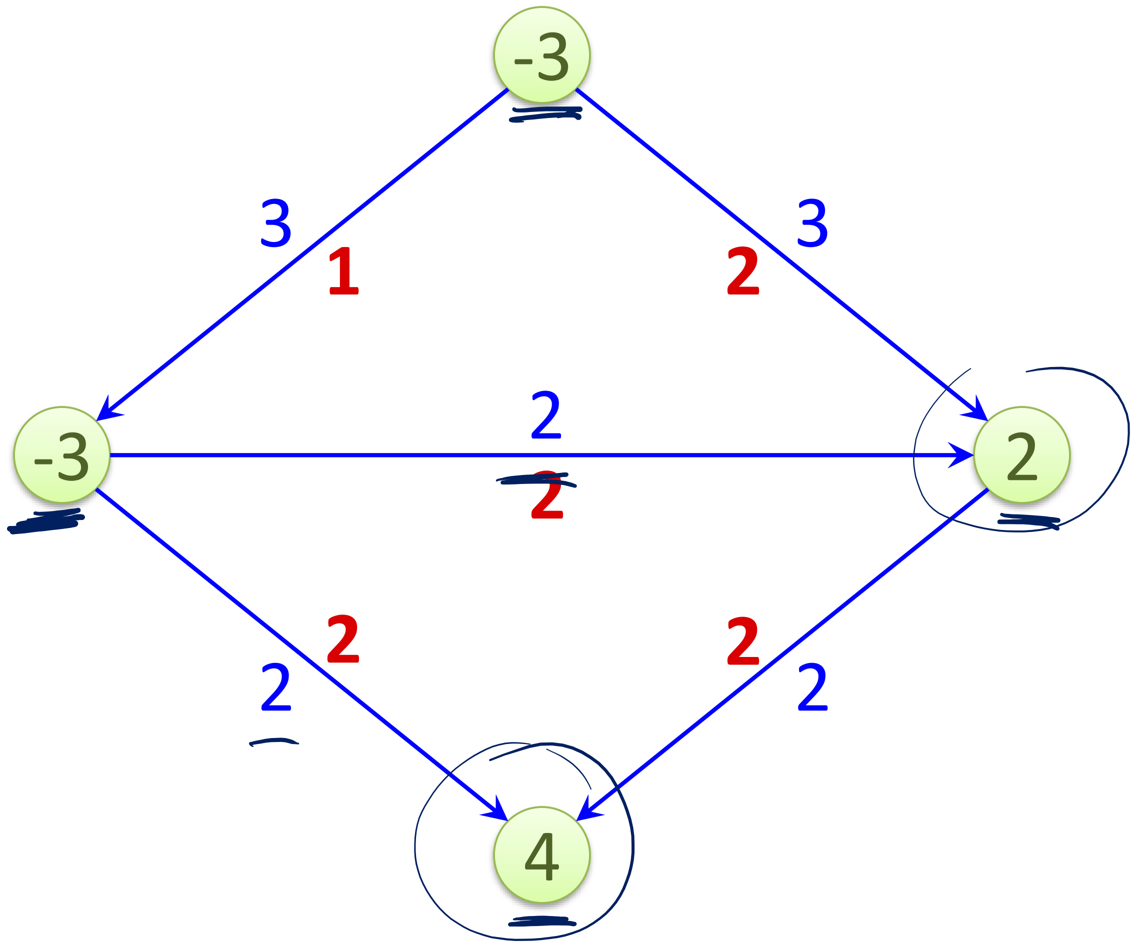
- Edge capacities $c_e > 0$ for all $e \in E$
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

Flow: Function $f: E \to \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E$: $\quad 0 \leq f(e) \leq c_e$
- *Demand Conditions*: $\forall v \in V$: $\quad f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f.

Example



Condition on Demands

Claim: If there exists a feasible circulation with demands d_v for $v \in V$, then

$$\sum_{v \in V} d_v = 0.$$

$$d_v = f^{in}(v) - f^{out}(v)$$

Proof:

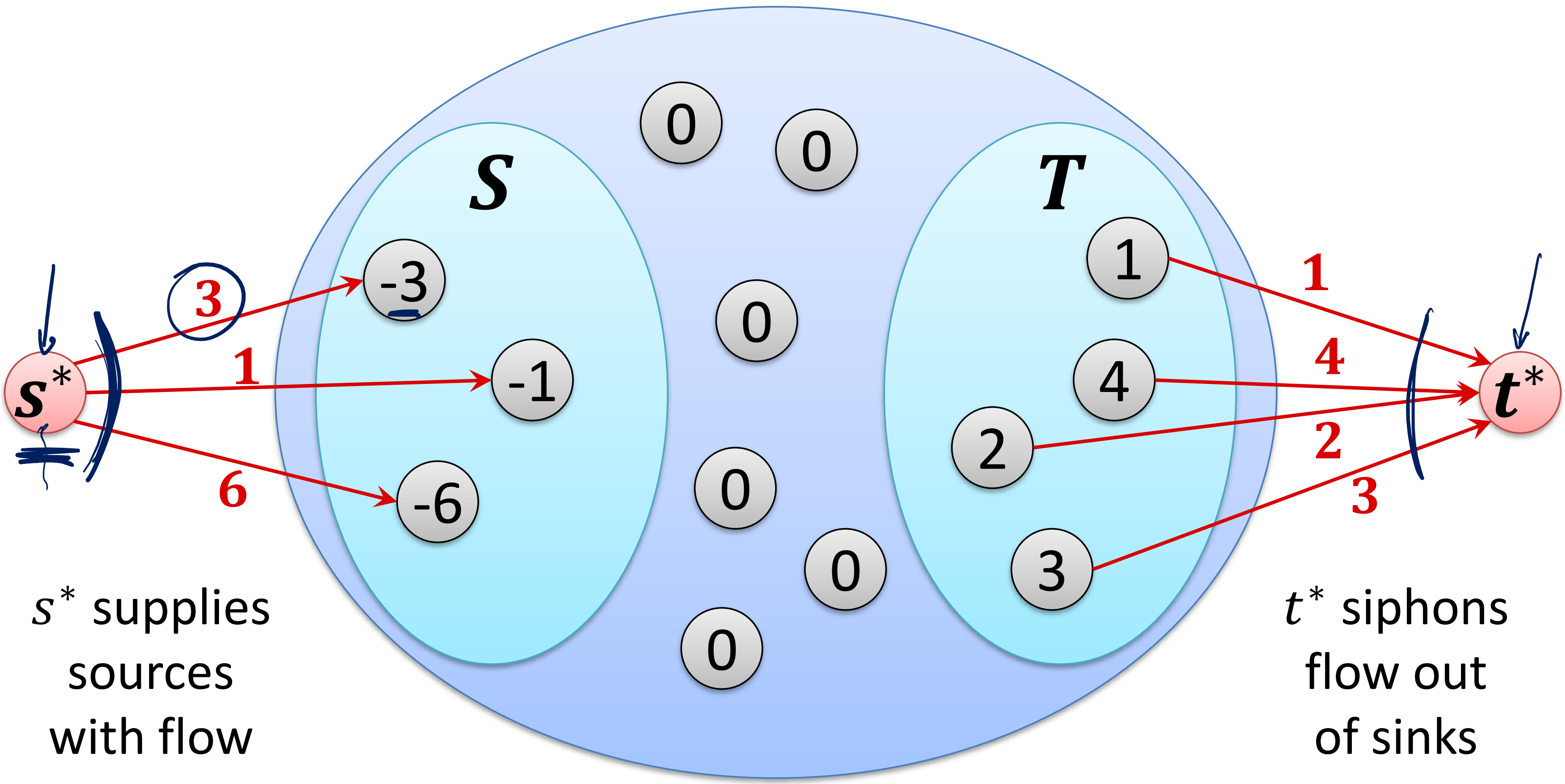
- $\sum_v d_v = \sum_v \left(f^{\text{in}}(v) - f^{\text{out}}(v)\right) = \sum_v f^{in}(v) - \sum_v f^{out}(v) = 0$
- $f(e)$ of each edge e appears twice in the above sum with different signs → overall sum is 0

Total supply = total demand:

$$\text{Define } \boldsymbol{D} \coloneqq \sum_{v:d_v>0} \boldsymbol{d_v} = \sum_{v:d_v<0} -\boldsymbol{d_v}$$

Reduction to Maximum Flow

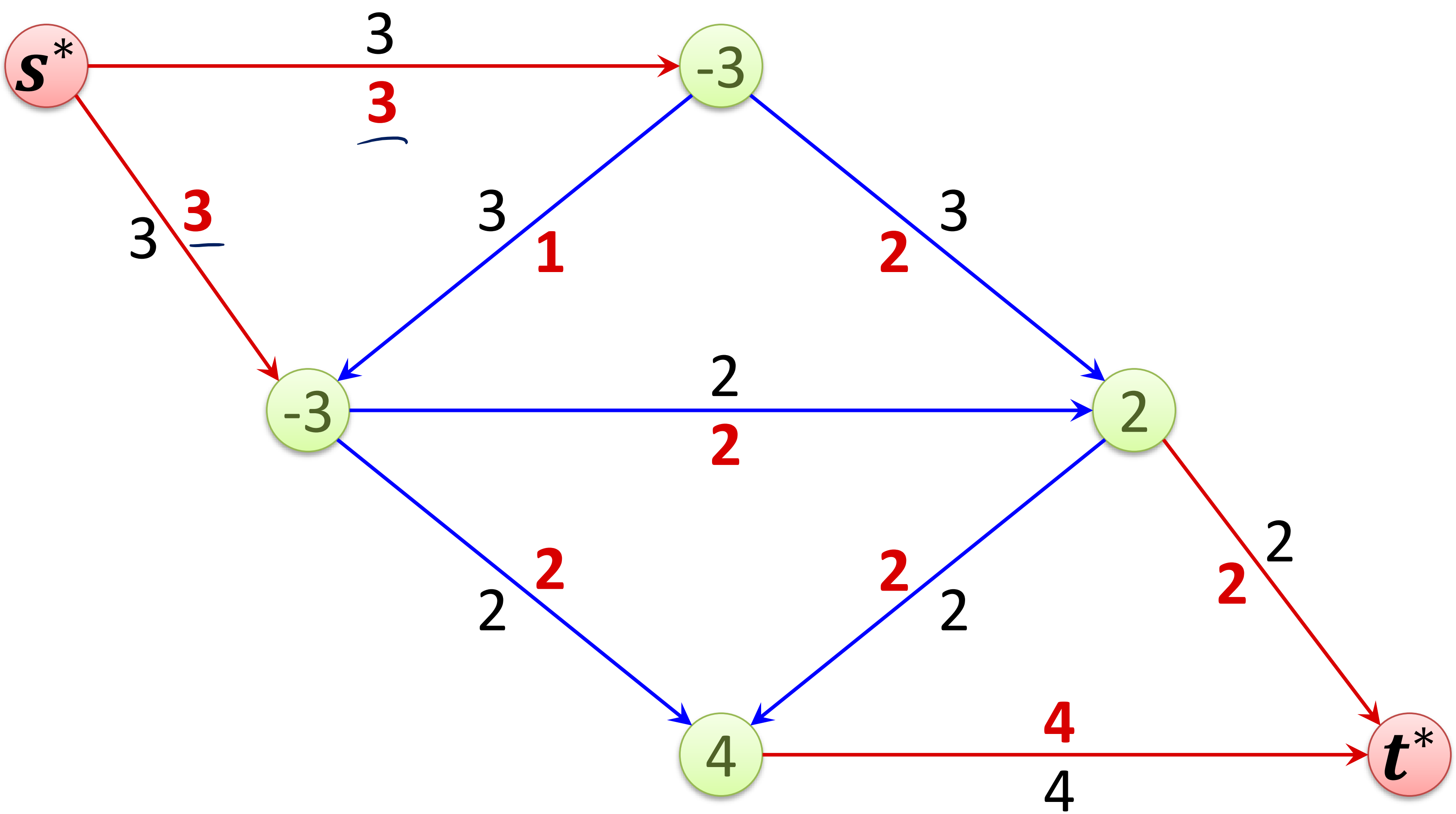
- Add “super-source” s^* and “super-sink” t^* to network



s^* supplies sources with flow

t^* siphons flow out of sinks

Example



Formally…

Reduction: Get graph G' from graph as follows

- Node set of G' is $V \cup \{s^*, t^*\}$
- Edge set is E and edges
  - (s^*, v) for all v with $d_v < 0$, capacity of edge is $-d_v$
  - (v, t^*) for all v with $d_v > 0$, capacity of edge is d_v

Observations:

- Capacity of min s^*-t^* cut is at most D (e.g., the cut $(s^*, V \cup \{t^*\})$)
- A feasible circulation on G can be turned into a feasible flow of value D of G' by saturating all (s^*, v) and (v, t^*) edges.
- Any flow of G' of value D induces a feasible circulation on G
  - (s^*, v) and (v, t^*) edges are saturated
  - By removing these edges, we get exactly the demand constraints

Circulation with Demands

Theorem: There is a feasible circulation with demands d_v, $v \in V$ on graph G if and only if there is a flow of value D on G'.

- If all capacities and demands are integers, there is an integer circulation

The max flow min cut theorem also implies the following:

Theorem: The graph G has a feasible circulation with demands d_v, $v \in V$ if and only if for all cuts (A, B),

$$\sum_{v \in B} d_v \leq c(A, B).$$