Chapter 7
Randomization

Algorithm Theory
WS 2019/20

Fabian Kuhn

Minimum Cut – Randomized Contraction Alg.

Reminder: Given a graph $G = (V, E)$, a cut is a partition (A, B) of V such that $V = A \cup B$, $A \cap B = \emptyset$, $A, B \neq \emptyset$

Size of the cut (A, B): # of edges crossing the cut

- For weighted graphs, total edge weight crossing the cut

Goal: Find a cut of minimal size (i.e., of size $\underline{\lambda(G)}$)

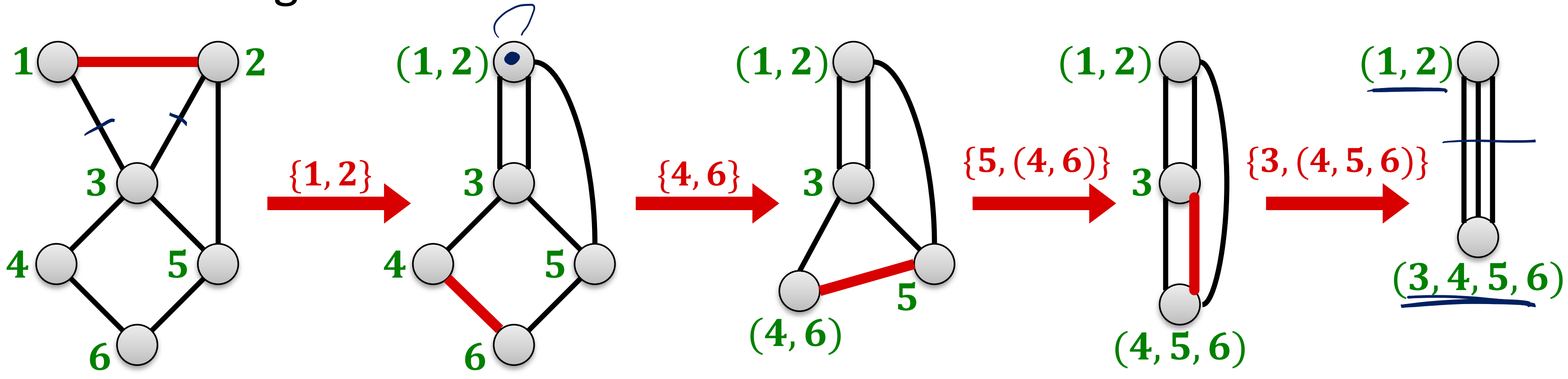
Basic randomized contraction algorithm:

```
while there are $> 2$ nodes do
    contract a uniformly random edge
return cut induced by the last two remaining nodes
        (cut defined by the original node sets represented by the last 2 nodes)
```

Edge Contractions

Nodes:

- After contracting $\{u, v\}$, the new node represents u and v
- After a series of contractions, each node represents a subset of the original nodes



Cuts:

- Assume in the contracted graph, w represents nodes $S_w \subset V$
- The edges of a node w in a contracted graph are in a one-to-one correspondence with the edges crossing the cut $(S_w, V \setminus S_w)$

Getting The Min Cut

specific

Theorem: The probability that the algorithm outputs a minimum cut is at least $2/(n(n-1))$.

(A,B)

To prove the theorem, we need the following claim:

Claim: If the minimum cut size of a multigraph G (no self-loops) is k, G has at least $kn/2$ edges.

Proof:

- Min cut has size $k \Longrightarrow$ all nodes have degree $\geq k$
  - A node v of degree $< k$ gives a cut $(\{v\}, V \setminus \{v\})$ of size $< k$
- Number of edges $m = {}^1\!/_2 \cdot \sum_v \deg(v)$

Better Randomized Algorithm

Let's simplify a bit:

- Pretend that $n/\sqrt{2}$ is an integer (for all n we will need it).
- Assume that a given min cut survives the first $n - {}^{n}/_{\sqrt{2}}$ contractions with probability $\geq {}^{1}/_{2}$.

$\mathbf{contract}(\boldsymbol{G}, \boldsymbol{t})$**:**

- Starting with n-node graph G, perform $n - t$ edge contractions such that the new graph has t nodes.

$\mathbf{mincut}(\boldsymbol{G})$**:**

```
1.  $X_1 := \mathrm{mincut}\big(\mathrm{contract}(G, n/\sqrt{2})\big)$;
2.  $X_2 := \mathrm{mincut}\big(\mathrm{contract}(G, n/\sqrt{2})\big)$;
3.  return $\min\{X_1, X_2\}$;
```

Running Time

Theorem: The running time of the recursive, randomized min cut algorithm is $O(n^2 \log n)$.

Proof:

- Can be shown in the usual way, by induction on n

Remark:

- The running time is only by an $O(\log n)$-factor slower than the basic contraction algorithm.
- The success probability is exponentially better!

Number of Minimum Cuts

- Given a graph G, how many minimum cuts can there be?

- Or alternatively: If G has edge connectivity $k \geq 1$, how many ways are there to remove k edges to disconnect G?

- Note that the total number of cuts is large.

n nodes

cut: partition of the nodes into 2 non-empty parts

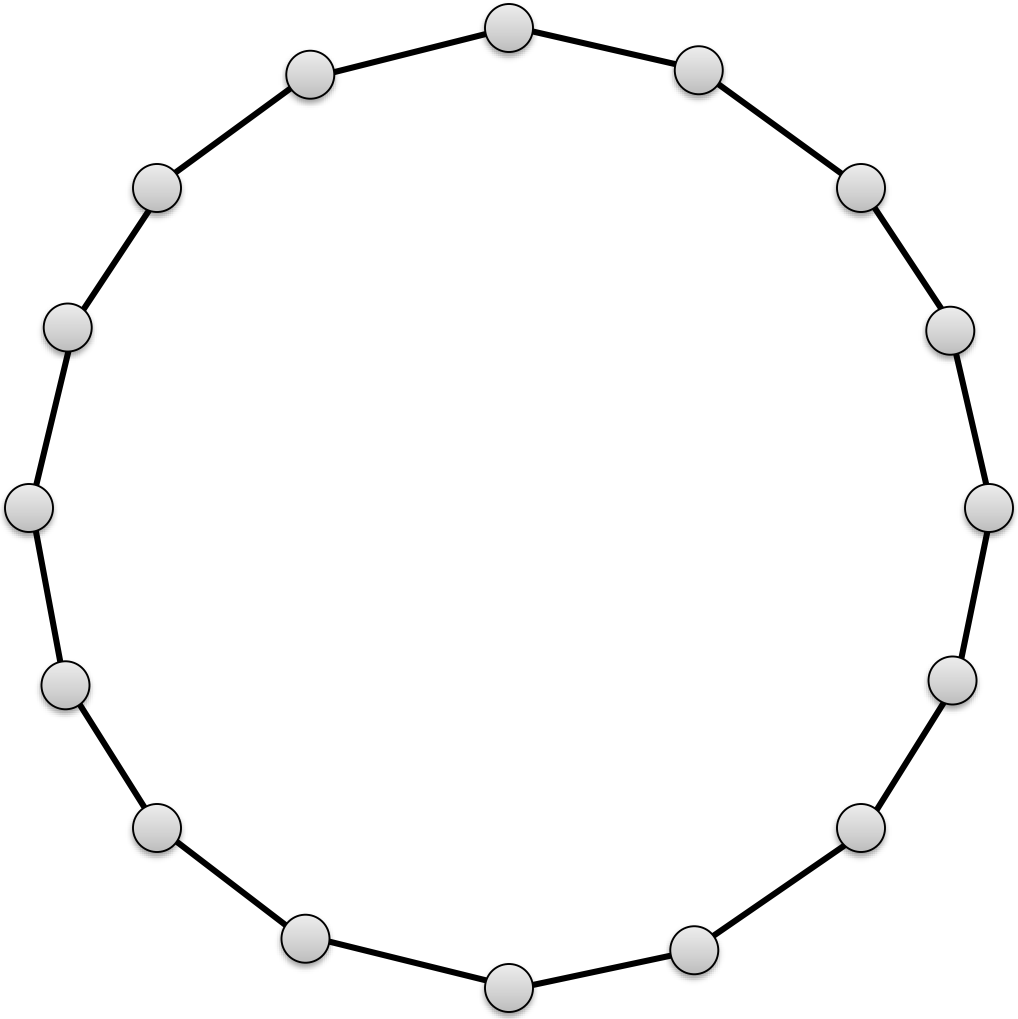
how many cuts are there? $2^{n-1} - 1$

$(A, V \setminus A)$ $(\emptyset, V)$

$(V \setminus A, A)$ $(V, \emptyset)$

Number of Minimum Cuts

Example: Ring with n nodes



- Minimum cut size: 2
- Every two edges induce a min cut
- Number of edge pairs:

  $\binom{n}{2}$

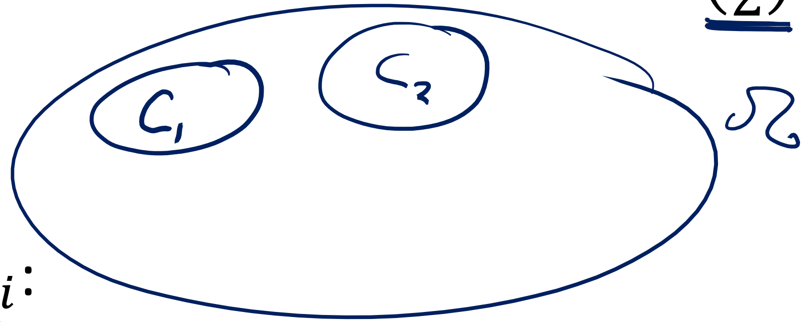
- Are there graphs with more min cuts?

Number of Min Cuts

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Theorem: The number of minimum cuts of a graph is at most $\underline{\binom{n}{2}}$.

Proof:



- Assume there are s min cuts
- For $i \in \{1, \dots, s\}$, define event $\mathcal{C}_i$:

$$\mathcal{C}_i := \{\text{basic contraction algorithm returns min cut } i\}$$

- We know that for $i \in \{1, \dots, s\}$: $\mathbb{P}(\mathcal{C}_i) \geq 1/\binom{n}{2} = \frac{2}{n(n-1)}$
- Events $\mathcal{C}_1, \dots, \mathcal{C}_s$ are disjoint:

$$1 \geq \mathbb{P}\left(\bigcup_{i=1}^{s} \mathcal{C}_i\right) = \sum_{i=1}^{s} \mathbb{P}(\mathcal{C}_i) \geq \frac{s}{\binom{n}{2}}$$

$$\frac{s}{\binom{n}{2}} \leq 1$$

$$\Updownarrow$$

$$s \leq \binom{n}{2}$$

Counting Larger Cuts

- In the following, assume that min cut has size $\lambda = \lambda(G)$
- How many cuts of size $\leq k = \alpha \cdot \lambda$ can a graph have?
- Consider a specific cut (A, B) of size $\leq k$
- As before, during the contraction algorithm:
  - min cut size $\geq \lambda$
  - total number of edges $\geq \lambda \cdot \#\text{nodes}/2$
  - cut (A, B) remains as long as none of its edges gets contracted

$n-(i-1) = n-i+1$

- Prob. that an edge crossing (A, B) is chosen in i^{th} contraction

$$\leq \frac{k}{\#\text{edges}} \leq \frac{2k}{\lambda \cdot \#\text{nodes}} = \frac{2\alpha}{n-i+1}$$

For simplicity, in the following, assume that 2α is an integer

Counting Larger Cuts

$1-\mathbb{P}(\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}) \le \frac{2\alpha}{n-i+1}$

Lemma: If $2\alpha \in \mathbb{N}$, the probability that cut (A, B) of size $\overset{\le}{\alpha \cdot \lambda}$ survives the first $n - 2\alpha$ edge contractions is at least

$$\frac{1}{\binom{n}{2\alpha}} = \frac{(2\alpha)!}{n(n-1)\cdot \ldots \cdot (n-2\alpha+1)} \ge \frac{2^{2\alpha-1}}{n^{2\alpha}}.$$

Proof:

- As before, event $\mathcal{E}_i$: cut (A, B) survives contraction i

$\mathbb{P}(\mathcal{E}_1)\cdot\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1)\cdot\mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_1 \cap \mathcal{E}_2)\cdot\ldots$

$$\ge \frac{\cancel{n-2\alpha}}{n}\cdot\frac{\cancel{n-2\alpha-1}}{n-1}\cdot\frac{n-2\alpha-2}{n-2}\cdot\ldots\cdot\frac{2}{2\alpha+2}\cdot\frac{1}{2\alpha+1}$$

$$= \frac{1\cdot 2\cdot\ldots\cdot 2\alpha}{n(n-1)\cdots(n-2\alpha+1)} = \frac{(2\alpha)!}{n(n-1)\cdots(n-2\alpha+1)} = \frac{1}{\binom{n}{2\alpha}}$$

Number of Cuts

Theorem: If $\underline{2\alpha \in \mathbb{N}}$, the number of edge cuts of size at most $\alpha \cdot \underline{\lambda(G)}$ in an n-node graph G is at most $\underline{n^{2\alpha}}$.

Proof:

of size $\leq \alpha \cdot \lambda$

$$\mathbb{P}(\text{cut } (A,B) \text{ survives } n-2\alpha \text{ edge contr.}) \geq \frac{2^{2\alpha-1}}{n^{2\alpha}}$$

$\underline{2\alpha \text{ nodes}}$: $\#\text{cuts} \leq 2^{2\alpha-1}$

return a random remaining cut

$$\mathbb{P}(\text{return } (A,B)) \geq \frac{2^{2\alpha-1}}{n^{2\alpha}} \cdot \frac{1}{\#\text{rem. cuts}}$$

$$= \frac{1}{n^{2\alpha}}$$

$$\Rightarrow \leq n^{2\alpha} \text{ cuts of size } \leq \alpha \cdot \lambda$$

Remark: The bound also holds for general $\alpha \geq 1$.

Resilience To Edge Failures

- Consider a network (a graph) G with n nodes

- Assume that each link (edge) of G fails independently with probability $\underline{p}$

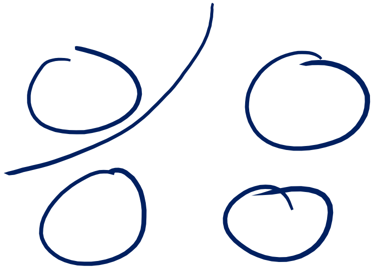
- How large can p be such that the remaining graph is still connected with high probability or with probability $1 - \varepsilon$?

Maintaining Connectivity

- A graph $G = (V, E)$ is connected iff every edge cut (A, B) has size at least 1.

cut of size 0 → graph disconnected

graph disconnected → ≥ 2 connected comp.



- We need to make sure that every cut keeps at least 1 edge

Resilience to Edge Failures

- Consider an edge cut (A, B) of size $k = \alpha \cdot \lambda(G)$
- Assume that each edge fails with probability $p \leq 1 - \frac{c \cdot \ln n}{\lambda(G)}$
- Hence each edge survives with probability $q \geq \frac{c \cdot \ln n}{\lambda(G)}$
- Probability that no edge crossing (A, B) survives

$$p^k = p^{\alpha\lambda} \le \left(1 - \frac{c \ln n}{\lambda}\right)^{\alpha\lambda} \le e^{-\frac{c \ln n}{\lambda} \cdot \alpha\lambda} = \underline{n^{-c\alpha}}$$

$$1 + x \le e^x$$

Maintaining All Cuts of a Certain Size

- The number of cuts of size $\underline{k} = \underline{\underline{\alpha\lambda(G)}}$ is at most $\underline{n^{2\alpha}}$.

Claim: If each edge survives with probability $q \geq \frac{c \cdot \ln(n)}{\underline{\lambda(G)}}$, with probability at least $1 - \overset{n^{(2-c)\alpha}}{\cancel{n^{-2\alpha}}}$, for a given $k = \alpha\lambda(G)$, at least one edge of each cut of size k survives.

number cuts of size k from $1, \dots, t \leq n^{2\alpha}$

for $i \in \{1, \dots, t\}$: B_i : no edge of cut i survives

$$\mathbb{P}(B_i) \leq n^{-c\alpha}$$

$$\underline{\mathbb{P}(B_1 \cup \dots \cup B_t)} \underset{\substack{\uparrow \\ \text{union bound}}}{\leq} \sum_{i=1}^{t} \mathbb{P}(B_i) \leq t \cdot n^{-c\alpha}$$

$$\underline{\underline{\leq n^{(2-c)\alpha}}}$$

Maintaining Connectivity $\Omega(q\cdot\lambda)$

Theorem: If each edge of a graph G independently survives with probability at least $\frac{(d+4)\cdot\ln n}{\lambda(G)}$, the remaining graph is connected with probability at least $1-\frac{1}{n^d}$.

A_k : some cut of size k does not survive

last slide: $\mathbb{P}(A_k) \le n^{(2-c)\alpha} \le n^{2-c}$ $(\forall k \ge \lambda)$

$\mathbb{P}(\text{some cut does not survive}) = \mathbb{P}(A_\lambda \cup A_{\lambda+1} \cup \dots \cup A_{n^2/4})$

$$\overset{\text{union bound}}{\le} \mathbb{P}(A_\lambda) + \mathbb{P}(A_{\lambda+1}) + \dots + \mathbb{P}(A_{n^2/4})$$

$$\le \frac{n^2}{4}\cdot n^{2-c} \le n^{4-c} \underset{\uparrow}{=} n^{-d}$$

$c=d+4$

$$\left(1-\frac{1}{\lambda}\right)^{\lambda}$$