# Chapter 8
# Approximation Algorithms

## Algorithm Theory
## WS 2019/20

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# Metric TSP

**Input:**

- Set $V$ of $n$ nodes (points, cities, locations, sites)
- Distance function $d: V \times V \to \mathbb{R}$, i.e., $d(u, v)$ is dist from $u$ to $v$
- Distances define a metric on $V$:

$$d(u,v) = d(v,u) \geq 0, \qquad d(u,v) = 0 \Leftrightarrow u = v$$
$$\forall u, v, w \in V : d(u,v) \leq d(u,w) + d(w,v)$$

**Solution:**

- Ordering/permutation $v_1, v_2, \dots, v_n$ of the vertices
- Length of TSP path: $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$
- Length of TSP tour: $d(v_1, v_n) + \sum_{i=1}^{n-1} d(v_i, v_{i+1})$

**Goal:**

- Minimize length of TSP path or TSP tour

# Metric TSP

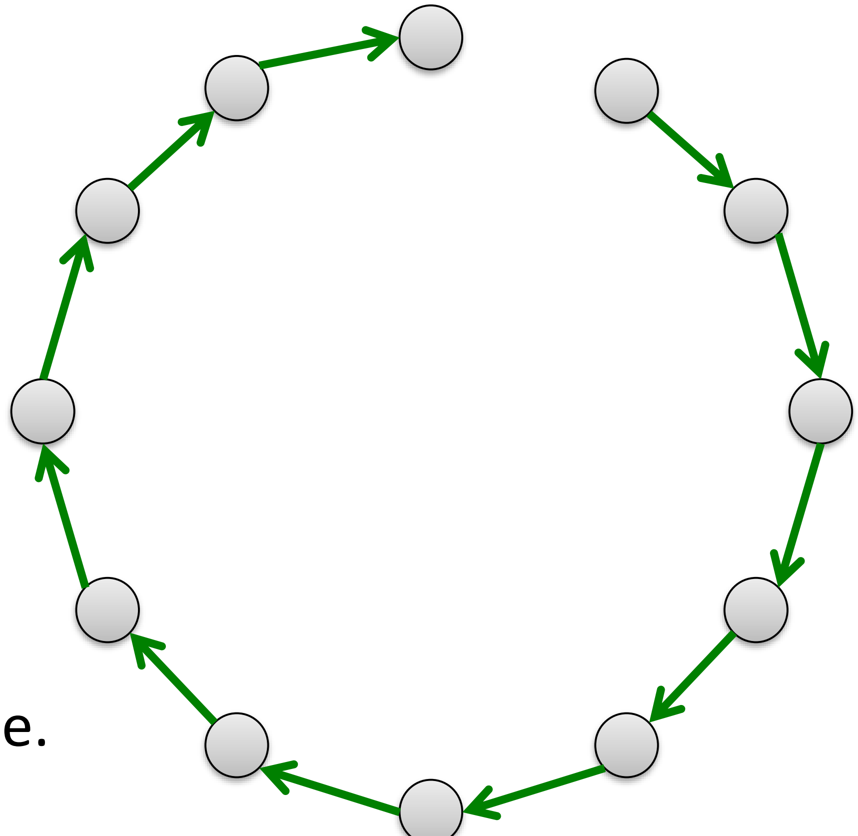
- The problem is NP-hard
- We have seen that the greedy algorithm (always going to the nearest unvisited node) gives an $O(\log n)$-approximation
- Can we get a constant approximation ratio?
- We will see that we can…

# TSP and MST

**Claim:** The length of an optimal TSP path is lower bounded by the weight of a minimum spanning tree

**Proof:**

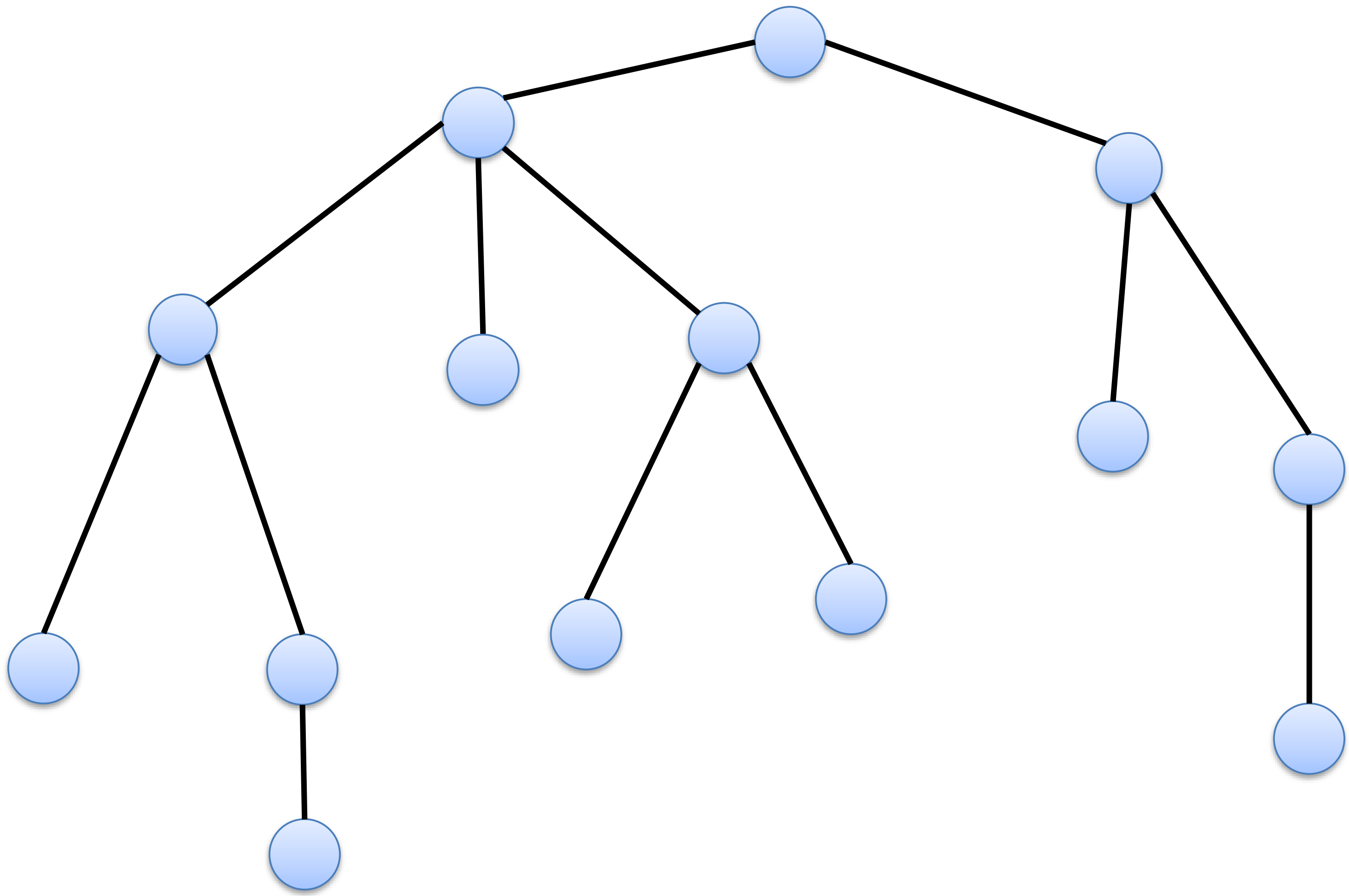
- A TSP path is a spanning tree, it's length is the weight of the tree



**Corollary:** Since an optimal TSP tour is longer than an optimal TSP path, the length of an optimal TSP tour is also lower bounded by the weight of a minimum spanning tree.

# The MST Tour

Walk around the MST…

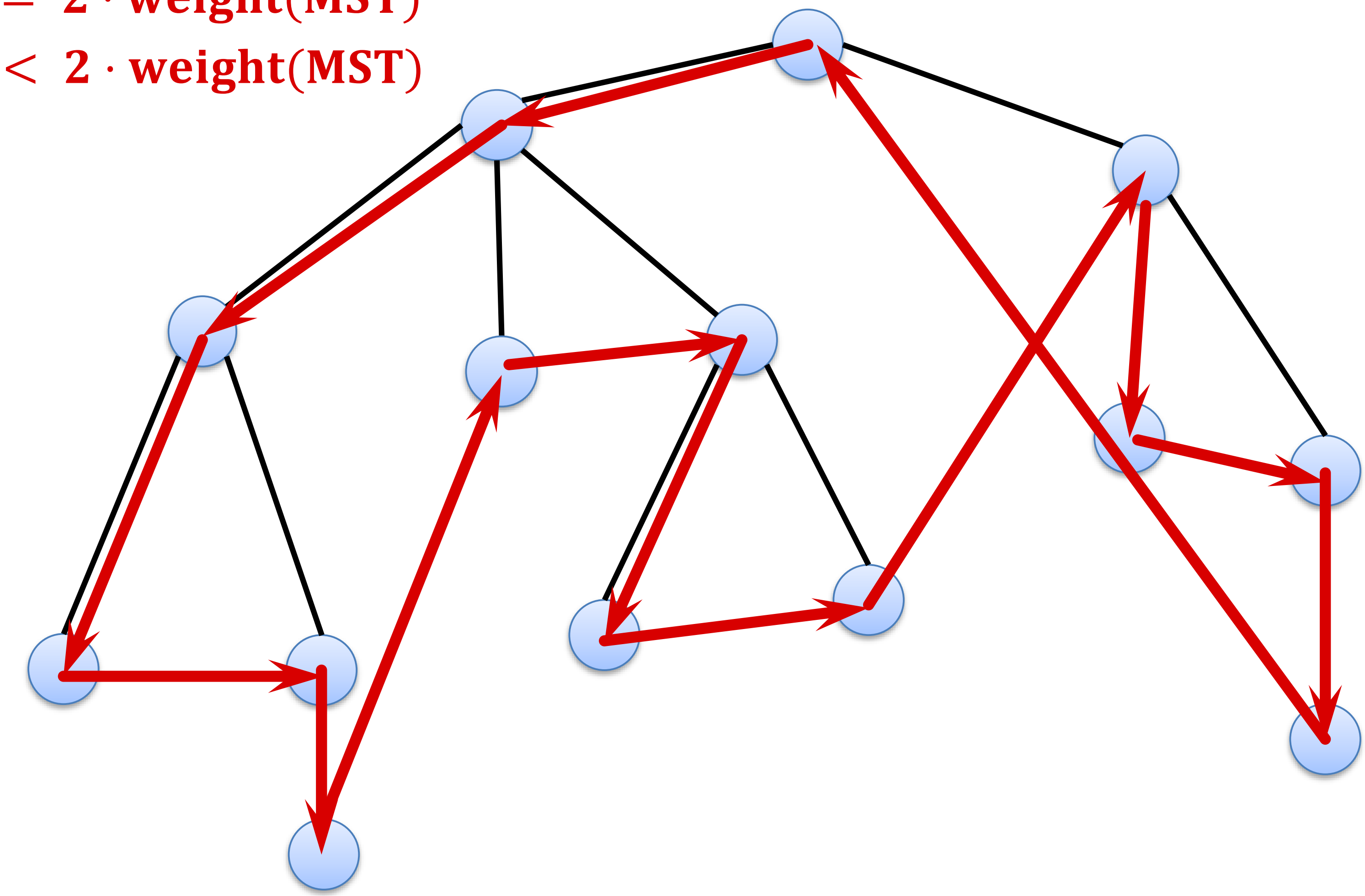


# The MST Tour

Walk around the MST…

**Cost (walk)** $= \ 2 \cdot \mathbf{weight(MST)}$

**Cost (tour)** $< \ 2 \cdot \mathbf{weight(MST)}$



# Approximation Ratio of MST Tour

**Theorem:** The MST TSP tour gives a 2-approximation for the metric TSP problem.

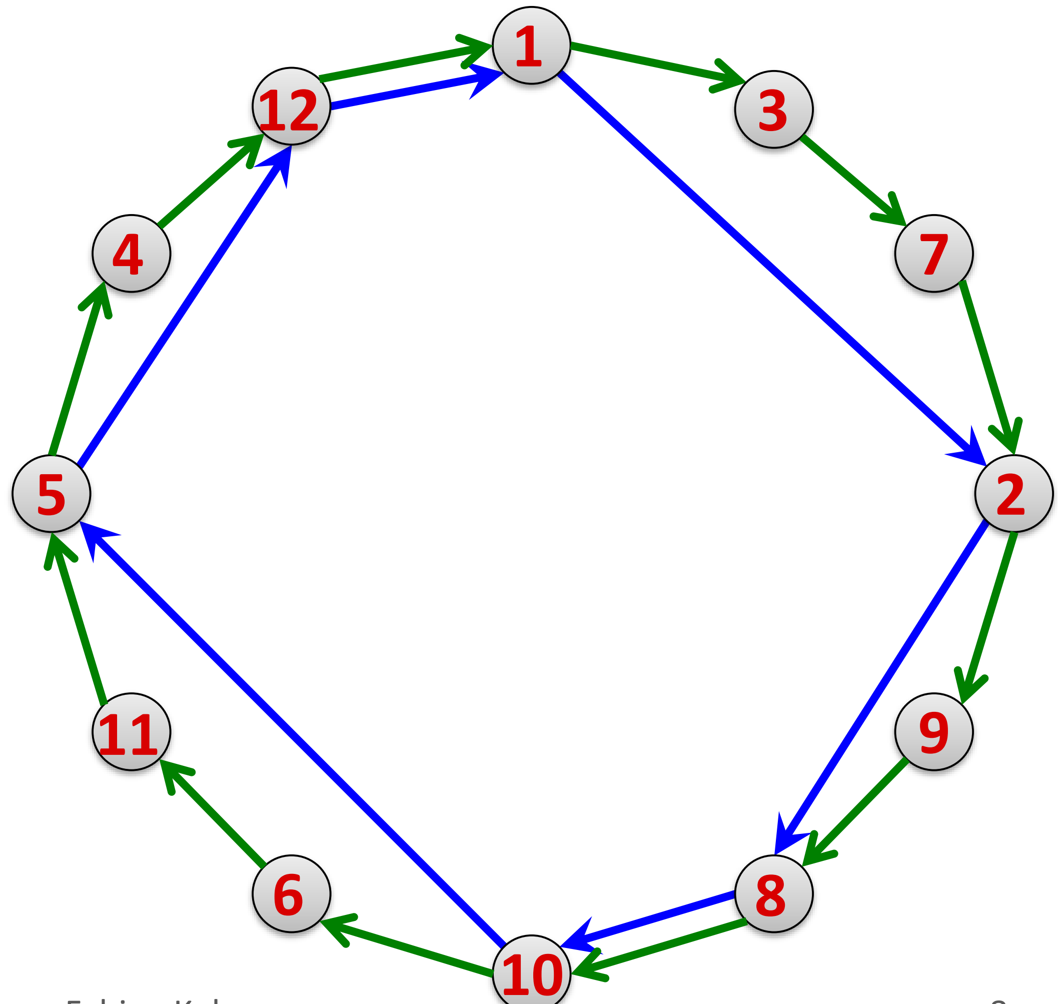
**Proof:**

- Triangle inequality → length of tour is at most $2 \cdot \text{weight}(\text{MST})$
- We have seen that $\text{weight}(\text{MST}) < \text{opt. tour length}$

Can we do even better?

# Metric TSP Subproblems

**Claim:** Given a metric $(V, d)$ and $(V', d)$ for $V' \subseteq V$, the optimal TSP path/tour of $(V', d)$ is at most as large as the optimal TSP path/tour of $(V, d)$.




**Optimal TSP tour of nodes $1, 2, \ldots, 12$**

**Induced TSP tour for nodes $1, 2, 5, 8, 10, 12$**

$\textbf{blue tour} \leq \textbf{green tour}$

# TSP and Matching

- Consider a metric TSP instance $(V, d)$ with an even number of nodes $|V|$

- Recall that a perfect matching is a matching $M \subseteq V \times V$ such that every node of $V$ is incident to an edge of $M$.

- Because $|V|$ is even and because in a metric TSP, there is an edge between any two nodes $u, v \in V$, any partition of $V$ into $|V|/2$ pairs is a perfect matching.

- The weight of a matching $M$ is the sum of the distances represented by all edges in $M$:

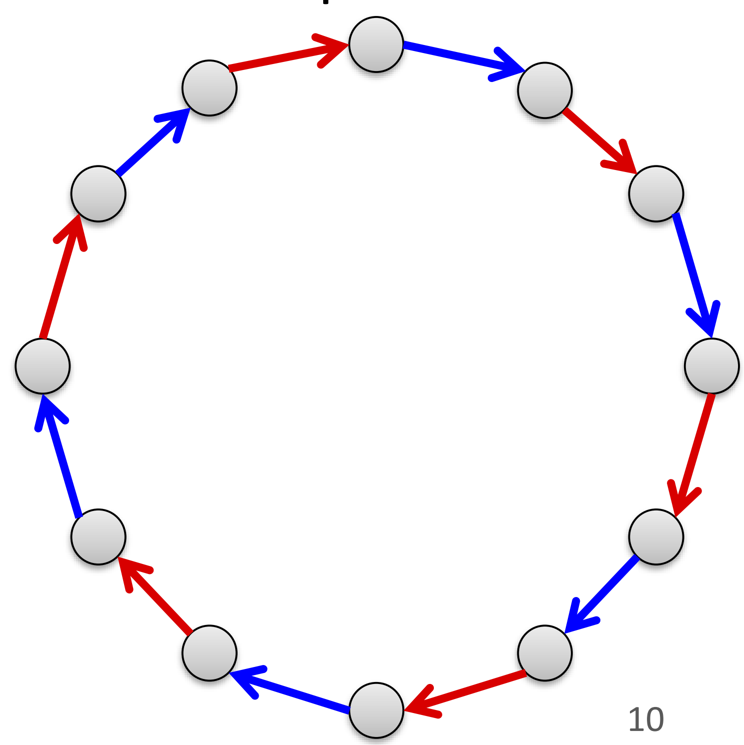
$$w(M) = \sum_{\{u,v\} \in M} d(u, v)$$

# TSP and Matching

**Lemma:** Assume we are given a TSP instance $(V, d)$ with an even number of nodes. The length of an optimal TSP tour of $(V, d)$ is at least twice the weight of a minimum weight perfect matching of $(V, d)$.

**Proof:**

- The edges of a TSP tour can be partitioned into 2 perfect matchings



# Minimum Weight Perfect Matching

**Claim:** If $|V|$ is even, a minimum weight perfect matching of $(V, d)$ can be computed in polynomial time

**Proof Sketch:**

- We have seen that a minimum weight perfect matching in a complete bipartite graph can be computed in polynomial time
- With a more complicated algorithm, also a minimum weight perfect matching in a complete (non-bipartite) graph can be computed in polynomial time
- The algorithm uses similar ideas as the bipartite weighted matching algorithm and it uses the Blossom algorithm as a subroutine

# Algorithm Outline

Problem of MST algorithm:

- Every edge has to be visited twice

**Goal:**

- Get a graph on which every edge only has to be visited once (and where still the total edge weight is small compared to an optimal TSP tour)

**Euler Tours:**

- A tour that visits each edge of a graph exactly once is called an Euler tour
- An Euler tour in a (multi-)graph exists if and only if every node of the graph has even degree
- That’s definitely not true for a tree, but can we modify our MST suitably?

# Euler Tour

**Theorem:** A connected (multi-)graph $G$ has an Euler tour if and only if every node of $G$ has even degree.

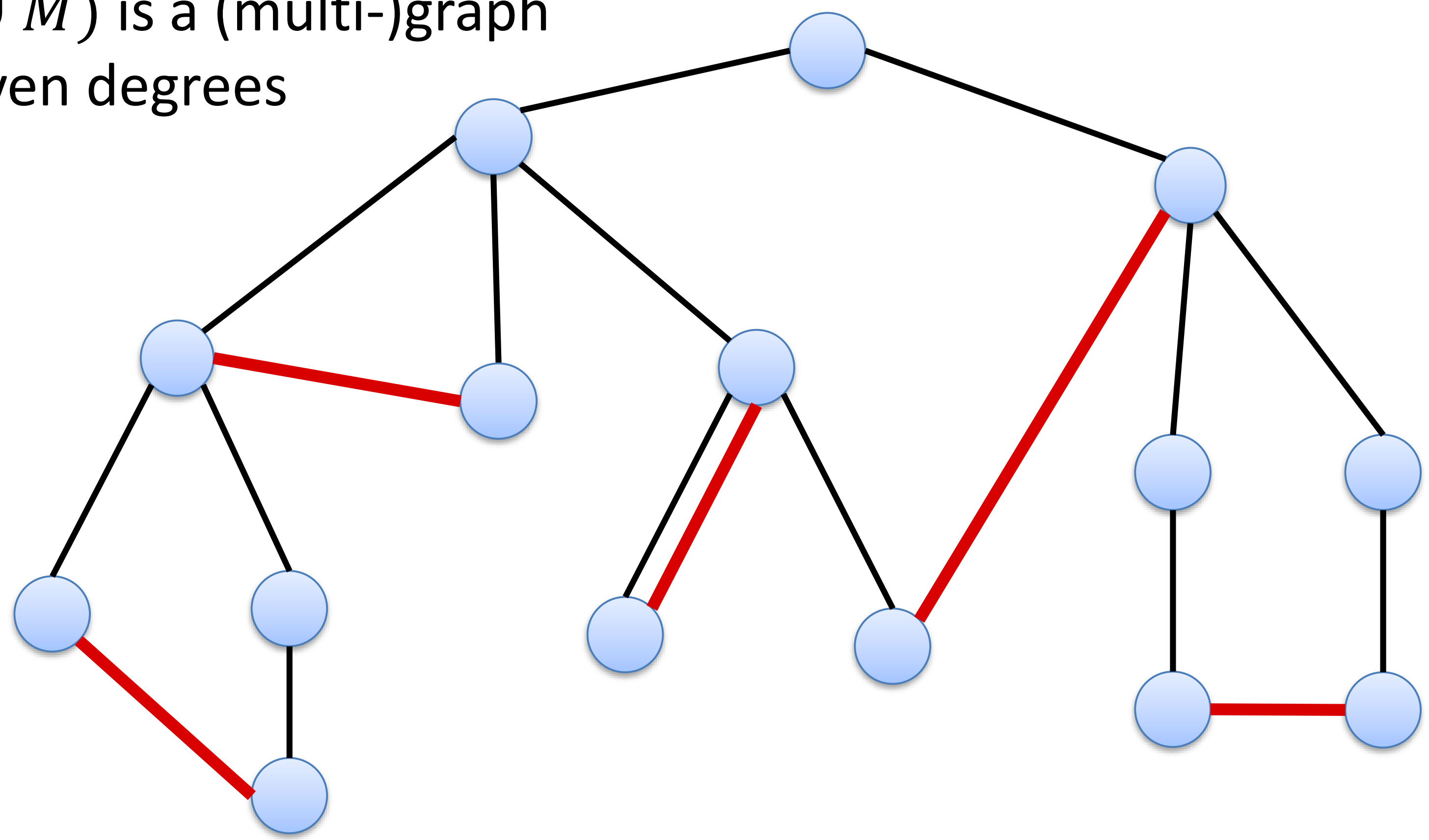
**Proof:**

- If $G$ has an odd degree node, it clearly cannot have an Euler tour
- If $G$ has only even degree nodes, a tour can be found recursively:

1. Start at some node
2. As long as possible, follow an unvisited edge
   - Gives a partial tour, the remaining graph still has even degree
3. Solve problem on remaining components recursively
4. Merge the obtained tours into one tour that visits all edges

# TSP Algorithm

1. Compute MST $T$
2. $V_{\mathrm{odd}}$: nodes that have an odd degree in $T$ ($|V_{\mathrm{odd}}|$ is even)
3. Compute min weight perfect matching $M$ of $(V_{\mathrm{odd}}, d)$
4. $(V, T \cup M)$ is a (multi-)graph with even degrees

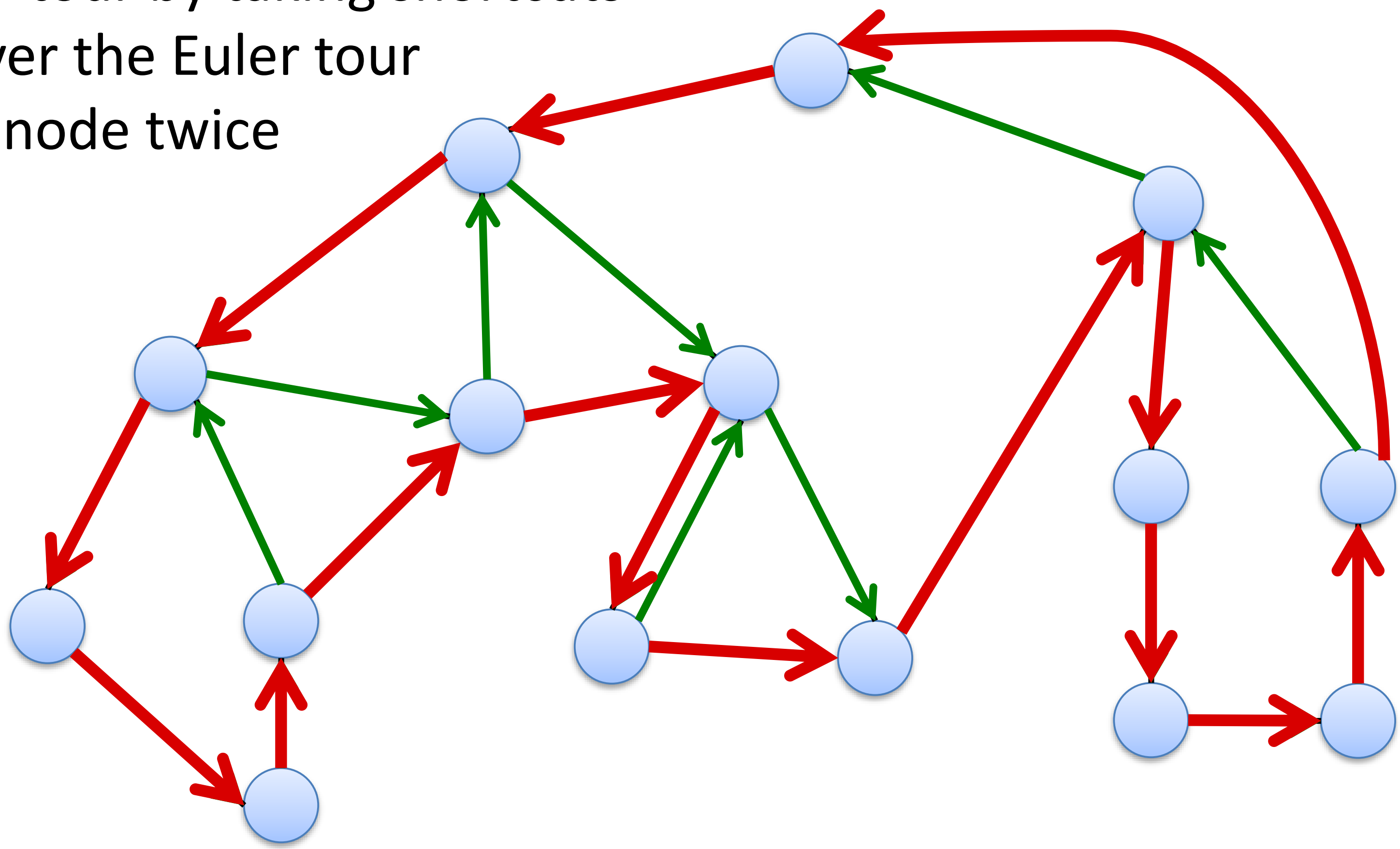


# TSP Algorithm

5. Compute Euler tour on $(V, T \cup M)$

6. Total length of Euler tour $\leq \frac{\mathbf{3}}{\mathbf{2}} \cdot \mathbf{TSP_{OPT}}$

7. Get TSP tour by taking shortcuts wherever the Euler tour visits a node twice



# TSP Algorithm

- The described algorithm is by Christofides

**Theorem:** The Christofides algorithm achieves an approximation ratio of at most ${}^{3}/_{2}$.

**Proof:**

- The length of the Euler tour is $\leq {}^{3}/_{2} \cdot \mathrm{TSP}_{\mathrm{OPT}}$
- Because of the triangle inequality, taking shortcuts can only make the tour shorter

# Set Cover

**Input:**

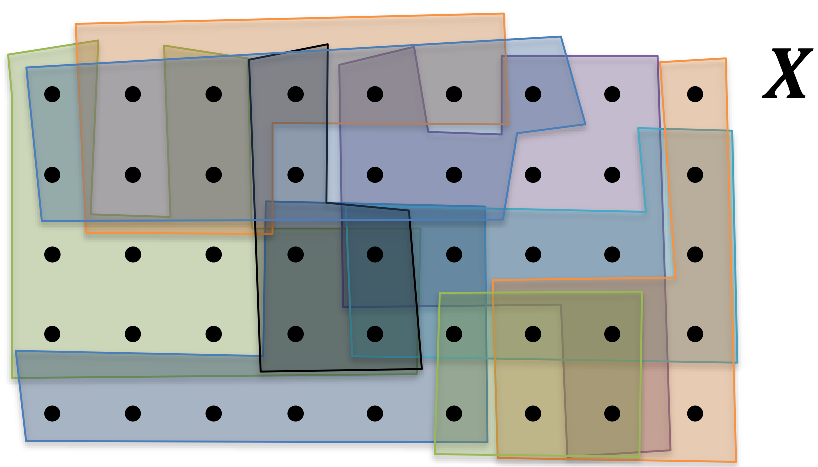
- A set of elements $X$ and a collection $\mathcal{S}$ of subsets $X$, i.e., $\mathcal{S} \subseteq 2^X$
  - such that $\bigcup_{S \in \mathcal{S}} S = X$

**Set Cover:**

- A set cover $\mathcal{C}$ of $(X, \mathcal{S})$ is a subset of the sets $\mathcal{S}$ which covers $X$:

$$\bigcup_{S \in \mathcal{C}} S = X$$

**Example:**



# Minimum (Weighted) Set Cover

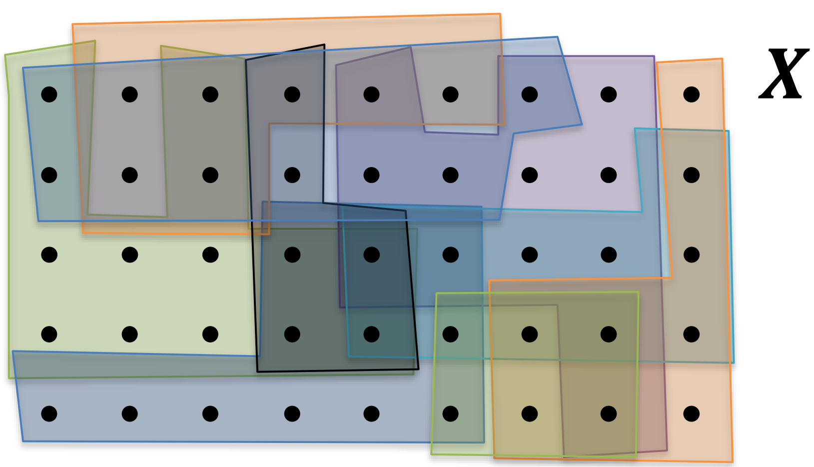
## Minimum Set Cover:

- **Goal:** Find a set cover $\mathcal{C}$ of smallest possible size
  - i.e., over $X$ with as few sets as possible

## Minimum Weighted Set Cover:

- Each set $S \in \mathcal{S}$ has a weight $w_S > 0$
- **Goal:** Find a set cover $\mathcal{C}$ of minimum weight

**Example:**

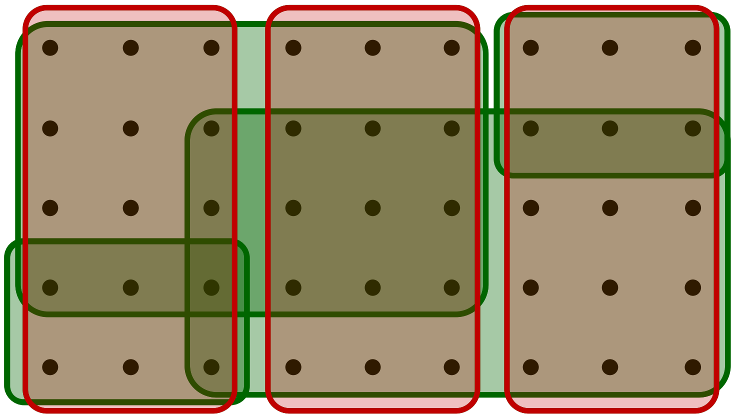


# Minimum Set Cover: Greedy Algorithm

**Greedy Set Cover Algorithm:**

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ to $\mathcal{C}$ s.t. $S$ covers as many uncovered elements as possible

**Example:**



# Weighted Set Cover: Greedy Algorithm

**Greedy Weighted Set Cover Algorithm:**

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ with the best weight per newly covered element ratio (set with best efficiency):

$$S = \arg\min_{S \in \mathcal{S} \setminus \mathcal{C}} \frac{w_S}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

**Analysis of Greedy Algorithm:**

- Assign a price $p(x)$ to each element $x \in X$:
  The efficiency of the set when covering the element
- If covering $x$ with set $S$, if partial cover is $\mathcal{C}$ before adding $S$:

$$p(e) = \frac{w_S}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

# Weighted Set Cover: Greedy Algorithm

**Example:**

- Universe $X = \{1,2,3,4,5,6,7,8,9,10\}$
- Sets $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

$$
\begin{aligned}
S_1 &= \{1,2,3,4,5\}, & w_{S_1} &= 4 \\
S_2 &= \{2,6,7\}, & w_{S_2} &= 1 \\
S_3 &= \{1,6,7,8,9\}, & w_{S_3} &= 4 \\
S_4 &= \{2,4,7,9,10\}, & w_{S_4} &= 6 \\
S_5 &= \{1,3,5,6,7,8,9,10\}, & w_{S_5} &= 9 \\
S_6 &= \{9,10\}, & w_{S_6} &= 3
\end{aligned}
$$

# Weighted Set Cover: Greedy Algorithm

**Lemma:** Consider a set $S = \{x_1, x_2, \dots, x_k\} \in \mathcal{S}$ be a set and assume that the elements are covered in the order $x_1, x_2, \dots, x_k$ by the greedy algorithm (ties broken arbitrarily).

Then, the price of element $x_i$ is at most $p(x_i) \leq \frac{w_S}{k-i+1}$

# Weighted Set Cover: Greedy Algorithm

**Lemma:** Consider a set $S = \{x_1, x_2, \dots, x_k\} \in \mathcal{S}$ be a set and assume that the elements are covered in the order $x_1, x_2, \dots, x_k$ by the greedy algorithm (ties broken arbitrarily).

Then, the price of element $x_i$ is at most $p(x_i) \leq \frac{w_S}{k-i+1}$

**Corollary:** The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{x \in S} p(x) \leq w_S \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

# Weighted Set Cover: Greedy Algorithm

**Corollary:** The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{x \in S} p(x) \leq w_S \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

**Theorem:** The approximation ratio of the greedy minimum (weighted) set cover algorithm is at most $\boldsymbol{H_s \leq 1 + \ln s}$, where $s$ is the cardinality of the largest set ($s = \max_{S \in \mathcal{S}} |S|$).

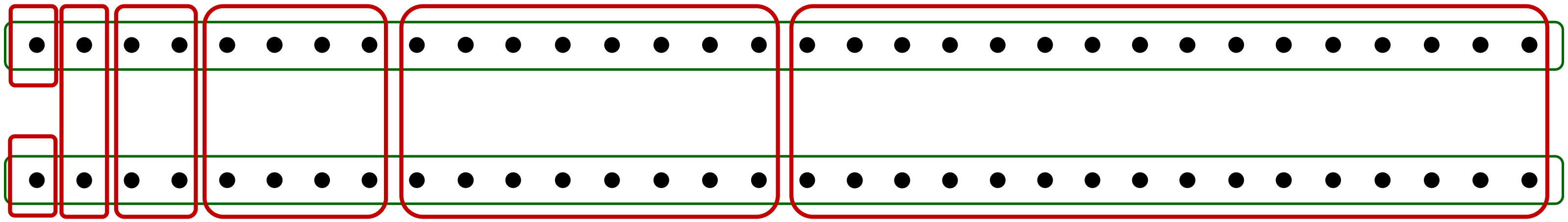
# Set Cover Greedy Algorithm

Can we improve this analysis?

No! Even for the unweighted minimum set cover problem, the approximation ratio of the greedy algorithm is $\geq \big(1 - o(1)\big) \cdot \ln s$.

- if $s$ is the size of the largest set... ($s$ can be linear in $n$)

Let's show that the approximation ratio is at least $\Omega(\log n)$...



$\mathbf{OPT = 2}$

$\mathbf{GREEDY \geq \log_2 n}$

# Set Cover: Better Algorithm?

An approximation ratio of $\ln n$ seems not spectacular...

Can we improve the approximation ratio?

No, unfortunately not, unless $\mathrm{P} = \mathrm{NP}$

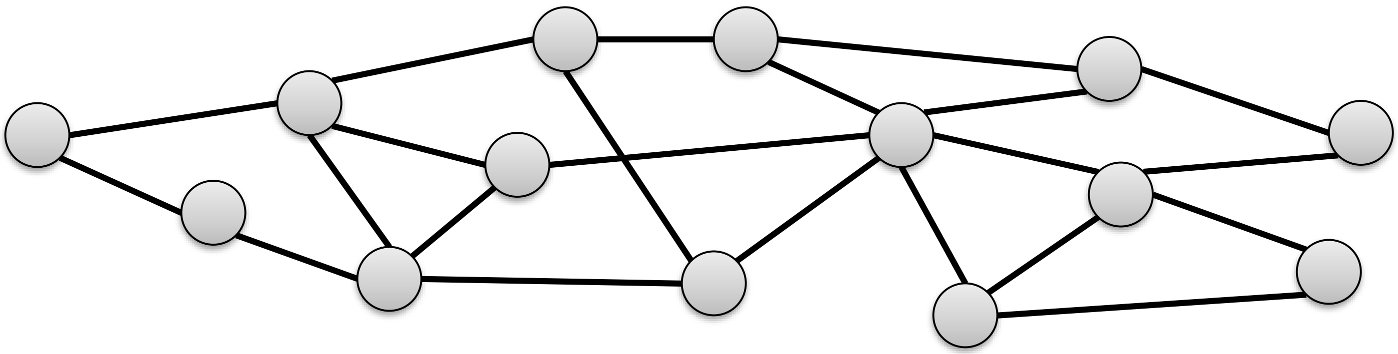
Dinur & Steurer showed in 2013 that unless $\mathrm{P} = \mathrm{NP}$, minimum set cover cannot be approximated better than by a factor $\big(1 - o(1)\big) \cdot \ln n$ in polynomial time.

- Proof is based on the so-called PCP theorem
  - PCP theorem is one of the main (relatively) recent advancements in theoretical computer science and the major tool to prove approximation hardness lower bounds
  - Shows that every language in NP has certificates of polynomial length that can be checked by a randomized algorithm by only querying a constant number of bits (for any constant error probability)

# Set Cover: Special Cases

**Vertex Cover:** set $S \subseteq V$ of nodes of a graph $G = (V, E)$ such that

$$\forall \{u, v\} \in E, \qquad \{u, v\} \cap S \neq \emptyset.$$



**Minimum Vertex Cover:**

- Find a vertex cover of minimum cardinality

**Minimum Weighted Vertex Cover:**

- Each node has a weight
- Find a vertex cover of minimum total weight

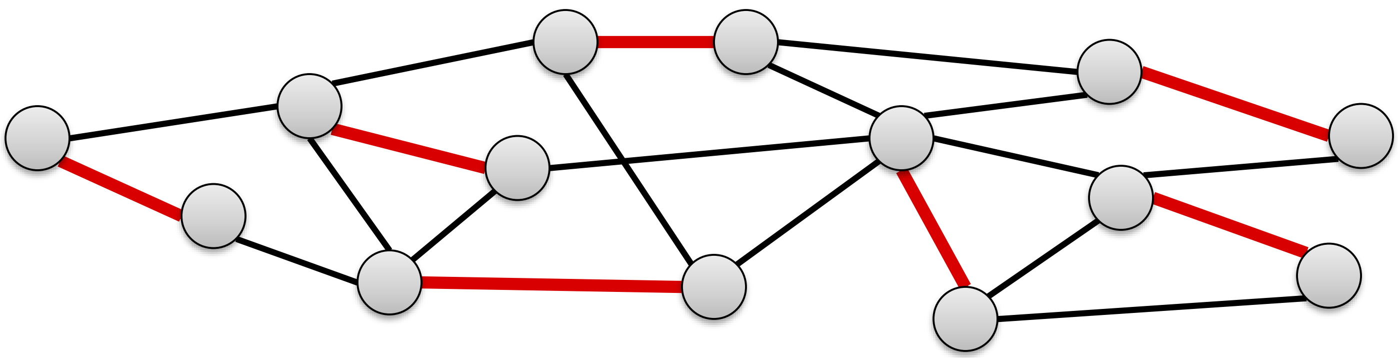
# Vertex Cover vs Matching

Consider a matching $M$ and a vertex cover $S$

**Claim:** $|M| \leq |S|$

**Proof:**

- At least one node of every edge $\{u, v\} \in M$ is in $S$
- Needs to be a different node for different edges from $M$



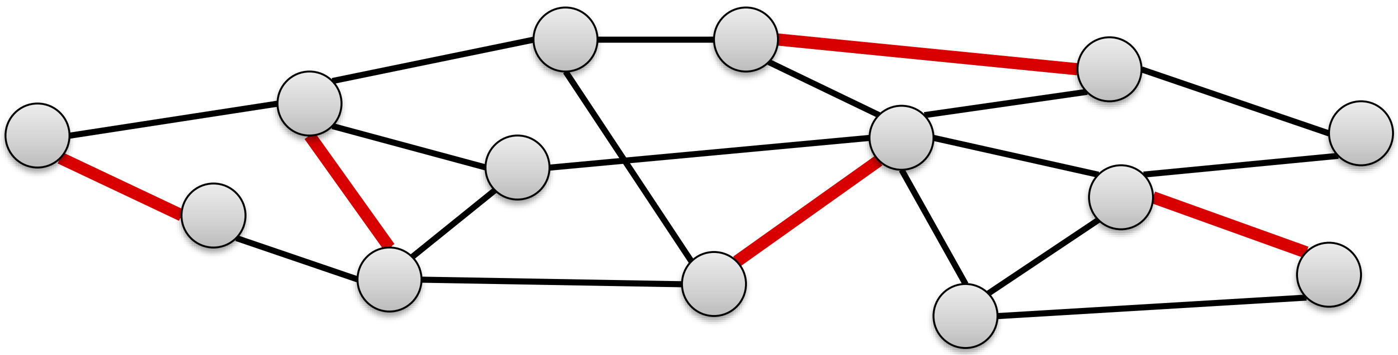
# Vertex Cover vs Matching

Consider a matching $M$ and a vertex cover $S$

**Claim:** If $M$ is maximal and $S$ is minimum, $|S| \leq 2|M|$

**Proof:**

- $M$ is maximal: for every edge $\{u, v\} \in E$, either $u$ or $v$ (or both) are matched



- Every edge $e \in E$ is "covered" by at least one matching edge
- Thus, the set of the nodes of all matching edges gives a vertex cover $S$ of size $|S| = 2|M|$.

# Maximal Matching Approximation

**Theorem:** For any maximal matching $M$ and any maximum matching $M^*$, it holds that

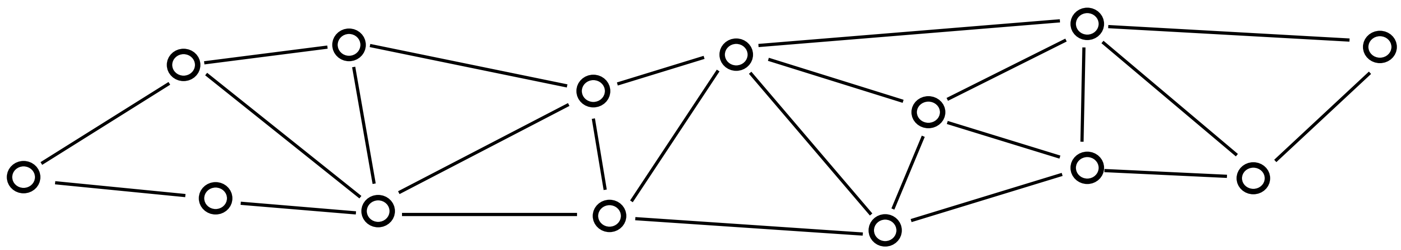
$$|M| \geq \frac{|M^*|}{2}.$$

**Proof:**

**Theorem:** The set of all matched nodes of a maximal matching $M$ is a vertex cover of size at most twice the size of a min. vertex cover.

# Set Cover: Special Cases

**Dominating Set:**
Given a graph $G = (V, E)$, a dominating set $S \subseteq V$ is a subset of the nodes $V$ of $G$ such that for all nodes $u \in V \setminus S$, there is a neighbor $v \in S$.



# Minimum Hitting Set

**Given:** Set of elements $X$ and collection of subsets $\mathcal{S} \subseteq 2^X$

– Sets cover $X$: $\bigcup_{S \in \mathcal{S}} S = X$

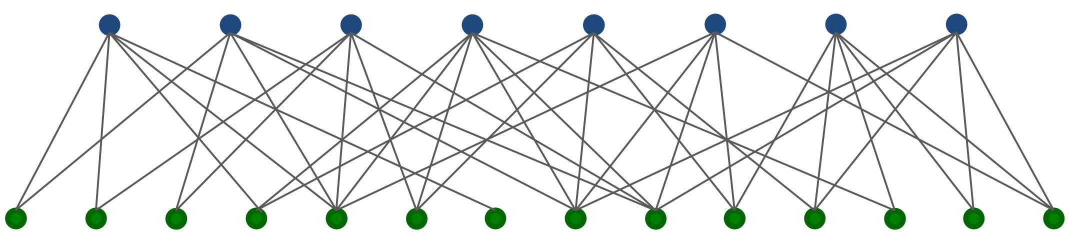
**Goal:** Find a min. cardinality subset $H \subseteq X$ of elements such that

$$\forall S \in \mathcal{S} : S \cap H \neq \emptyset$$

Problem is equivalent to min. set cover with roles of sets and elements interchanged

**Sets**

**Elements**



# Knapsack

- $n$ items $1, \dots, n$, each item has weight $w_i > 0$ and value $v_i > 0$
- Knapsack (bag) of capacity $W$

- Goal: pack items into knapsack such that total weight is at most $W$ and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t. } S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length $w_i$ and value $v_i$, server available for $W$ time units, try to execute a set of jobs that maximizes the total value

# Knapsack: Dynamic Programming Alg.

**We have shown:**

- If all item weights $w_i$ are integers, using dynamic programming, the knapsack problem can be solved in time $O(nW)$
- If all values $v_i$ are integers, there is another dynamic progr. algorithm that runs in time $O(n^2V)$, where $V$ is the max. value.