# Chapter 9
# Online Algorithms

## Algorithm Theory
## WS 2019/20

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# Online Computations

- Sometimes, an algorithm has to start processing the input before the complete input is known

- For example, when storing data in a data structure, the sequence of operations on the data structure is not known

**Online Algorithm:** An algorithm that has to produce the output step-by-step when new parts of the input become available.

**Offline Algorithm:** An algorithm that has access to the whole input before computing the output.

- Some problems are inherently online
  - Especially when real-time requests have to be processed over a significant period of time

# Competitive Ratio

- Let's again consider optimization problems
  - For simplicity, assume, we have a minimization problem

**Optimal offline solution $\mathbf{OPT}(\boldsymbol{I})$:**

- Best objective value that an offline algorithm can achieve for a given input sequence $I$

**Online solution $\mathbf{ALG}(\boldsymbol{I})$:**

- Objective value achieved by an online algorithm $\mathrm{ALG}$ on $I$

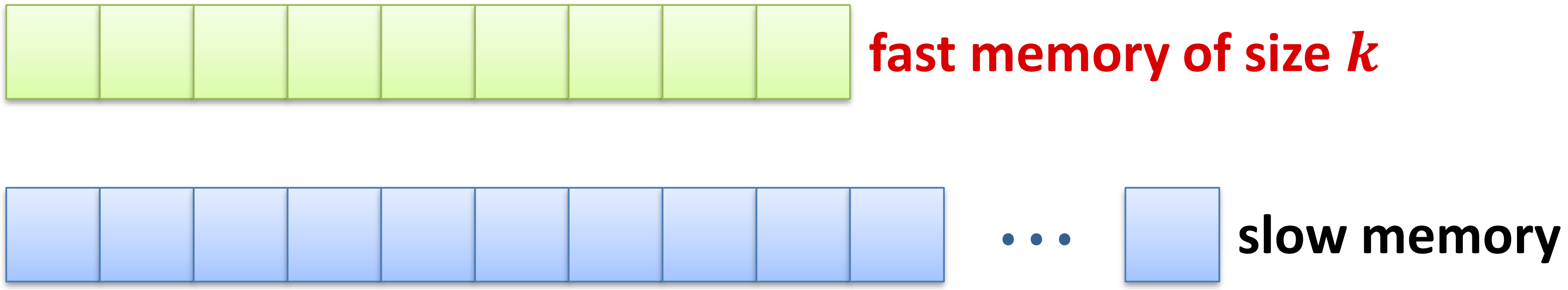
**Competitive Ratio:** An algorithm has competitive ratio $c \geq 1$ if

$$\mathbf{ALG}(\boldsymbol{I}) \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha = 0$, we say that $\mathrm{ALG}$ is strictly $c$-competitive.

# Paging Algorithm

Assume a simple memory hierarchy:



If a memory page has to be accessed:

- Page in fast memory (hit): take page from there
- Page not in fast memory (miss): leads to a page fault
- Page fault: the page is loaded into the fast memory and some page has to be evicted from the fast memory
- Paging algorithm: decides which page to evict
- Classical online problem: we don't know the future accesses

## Paging Strategies

**Least Recently Used (LRU):**

- Replace the page that hasn’t been used for the longest time

**First In First Out (FIFO):**

- Replace the page that has been in the fast memory longest

**Last In First Out (LIFO):**

- Replace the page most recently moved to fast memory

**Least Frequently Used (LFU):**

- Replace the page that has been used the least

**Longest Forward Distance (LFD):**

- Replace the page whose next request is latest (in the future)
- LFD is **not** an online strategy!

# LFD is Optimal

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

- For contradiction, assume that LFD is not optimal
- Then there exists a finite input sequence $\sigma$ on which LFD is not optimal (assume that the length of $\sigma$ is $|\sigma| = n$)
- Let OPT be an optimal solution for $\sigma$ such that
  - OPT processes requests $1, \dots, i$ in exactly the same way as LFD
  - OPT processes request $i + 1$ differently than LFD
  - Any other optimal strategy processes one of the first $i + 1$ requests differently than LFD
- Hence, OPT is the optimal solution that behaves in the same way as LFD for as long as possible → we have $i < n$
- Goal: Construct $\mathrm{OPT}'$ that is identical with LFD for req. $1, \dots, i + 1$

# LFD is Optimal

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

**Case 1:** Request $i + 1$ does **not** lead to a page fault

- LFD does not change the content of the fast memory
- OPT behaves differently than LFD
  → OPT replaces some page in the fast memory
  - As up to request $i + 1$, both algorithms behave in the same way, they also have the same fast memory content
  - OPT therefore does not require the new page for request $i + 1$
  - Hence, OPT can also load that page later (without extra cost) → $\mathrm{OPT}'$

# LFD is Optimal

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

**Case 2:** Request $i + 1$ does lead to a **page fault**

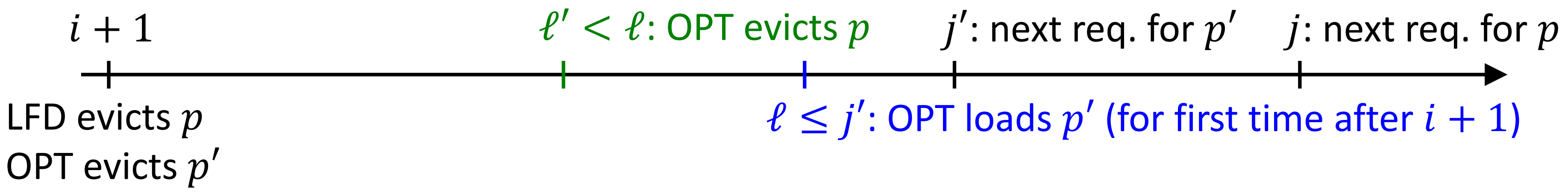
- LFD and OPT move the same page into the fast memory, but they evict different pages
  - If OPT loads more than one page, all pages that are not required for request $i + 1$ can also be loaded later
- Say, LFD evicts page $p$ and OPT evicts page $p'$
- By the definition of LFD, $p'$ is required again before page $p$

# LFD is Optimal

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

**Case 2:** Request $i + 1$ does lead to a **page fault**


$i + 1$
$\ell' < \ell$: OPT evicts $p$
$j'$: next req. for $p'$
$j$: next req. for $p$
LFD evicts $p$
OPT evicts $p'$
$\ell \leq j'$: OPT loads $p'$ (for first time after $i + 1$)

a) OPT keeps $p$ in fast memory until request $\ell$

– Evict $p$ at request $i + 1$, keep $p'$ instead and load $p$ (instead of $p'$) back into the fast memory at request $\ell$

b) OPT evicts $p$ at request $\ell' < \ell$

– Evict $p$ at request $i + 1$ and $p'$ at request $\ell'$ (switch evictions of $p$ and $p'$)

# Phase Partition

We partition a given request sequence $\sigma$ into phases as follows:

- **Phase $\mathbf{0}$**: empty sequence
- **Phase $\boldsymbol{i}$** : maximal sequence that immediately follows phase $i-1$ and contains at most $k$ distinct page requests

**Example sequence ($\boldsymbol{k = 4}$):**

$2, 5, 12, 5, 4, 2, 10, 8, 3, 6, 2, 2, 6, 6, 8, 3, 2, 6, 9, 10, 6, 3, 10, 2, 1, 3, 5$

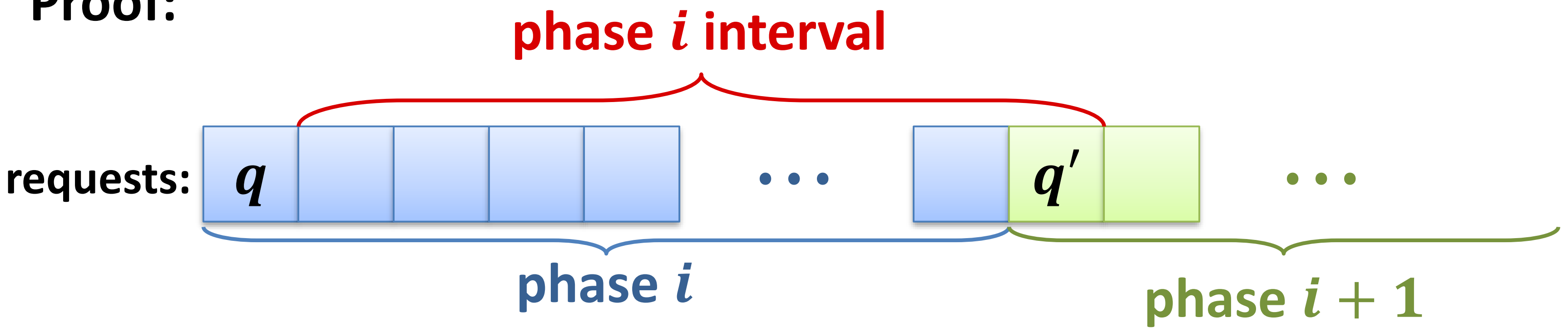
**Phase $\boldsymbol{i}$ Interval:** interval starting with the second request of phase $i$ and ending with the first request of phase $i+1$

- If the last phase is phase $p$, phase $i$ interval is defined for $i = 1, \dots, p-1$

## Optimal Algorithm

**Lemma:** Algorithm LFD has at least one page fault in each phase $i$ interval (for $i = 1, \dots, p-1$, where $p$ is the number of phases).

**Proof:**



- $q$ is in fast memory after first request of phase $i$
- Number of distinct requests in phase $i$: $k$
- By maximality of phase $i$: $q'$ does not occur in phase $i$
- Number of distinct requests $\neq q$ in phase interval $i$: $k$

→ at least one page fault

# LRU and FIFO Algorithms

**Lemma:** Algorithm LFD has at least one page fault in each phase $i$ interval (for $i = 1, \dots, p-1$, where $p$ is the number of phases).

**Corollary:** The number of page faults of an optimal offline algorithm is at least $p-1$, where $p$ is the number of phases

**Theorem:** The LRU and the FIFO algorithms both have a competitive ratio of at most $k$.

**Proof:**

- We will show that both have at most $k$ page faults per phase
- We then have (for every input $I$):

$$\mathrm{LRU}(I), \mathrm{FIFO}(I) \leq k \cdot p \leq k \cdot \mathrm{OPT}(I) + k$$

# LRU and FIFO Algorithms

**Theorem:** The LRU and the FIFO algorithms both have a competitive ratio of at most $k$.

**Proof:**

- Need to show that both have at most $k$ page faults per phase
- LRU:
  - The $k$ last pages used are the $k$ least recently used
  - Throughout a phase $i$, the $k$ distinct pages of phase $i$ are the l.r.u.
  - Once in the fast memory, these pages are therefore not evicted until the end of the phase
- FIFO:
  - In each page fault in phase $i$, one of the $k$ pages of phase $i$ is loaded into fast memory
  - Once a page is loaded in a page fault of phase $i$ it belongs to the least $k$ pages loaded into fast memory throughout the rest of the phase
  - Hence: Each of the $k$ pages leads to $\leq 1$ page fault in phase $i$

# Lower Bound

**Theorem:** Even if the slow memory contains only $k + 1$ pages, any deterministic algorithm has competitive ratio at least $k$.

**Proof:**

- Consider some given deterministic algorithm ALG
- Because ALG is deterministic, the content of the fast memory after the first $i$ requests is determined by the first $i$ requests.
- Construct a request sequence inductively as follows:
  - Assume some initial slow memory content
  - The $(i + 1)^{\text{st}}$ request is for the page which is not in fast memory after the first $i$ requests (throughout we only use $k + 1$ different pages)
- There is a page fault for every request
- OPT has a page fault at most every $k$ requests
  - There is always a page that is not required for the next $k - 1$ requests

# Randomized Algorithms

- We have seen that deterministic paging algorithms cannot be better than $k$-competitive

- Does it help to use randomization?

**Competitive Ratio:** A randomized online algorithm has competitive ratio $c \geq 1$ if for all inputs $I$,

$$\mathbb{E}[\mathbf{ALG}(\boldsymbol{I})] \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha \leq 0$, we say that $\mathrm{ALG}$ is strictly $c$-competitive.

# Adversaries

- For randomized algorithm, we need to distinguish between different kinds of adversaries (providing the input)

**Oblivious Adversary:**

- Has to determine the complete input sequence before the algorithm starts
  - The adversary cannot adapt to random decisions of the algorithm

**Adaptive Adversary:**

- The input sequence is constructed during the execution
- When determining the next input, the adversary knows how the algorithm reacted to the previous inputs
- Input sequence depends on the random behavior of the alg.
- Sometimes, two adaptive adversaries are distinguished
  - offline, online : different way of measuring the adversary cost

# Lower Bound

The adversaries can be ordered according to their strength

$$\text{oblivious} < \text{online adaptive} < \text{offline adaptive}$$

- An algorithm that achieves a given comp. ratio with an adaptive adversary is at least as good with an oblivious one
- A lower bound that holds against an oblivious adversary also holds for the two adaptive adversaries
- …

**Theorem:** No randomized paging algorithm can be better than $k$-competitive against an adaptive adversary.

**Proof:** The same proof as for deterministic algorithms works.

- Are there better algorithms with an oblivious adversary?

# The Randomized Marking Algorithm

- Every entry in fast memory has a marked flag

- Initially, all entries are unmarked.
- If a page in fast memory is accessed, it gets marked
- When a page fault occurs:
  - If all $k$ pages in fast memory are marked, all marked bits are set to $0$
  - The page to be evicted is chosen uniformly at random among the unmarked pages
  - The marked bit of the new page in fast memory is set to $1$

## Example

**Input Sequence (k=6):**

$$\underbrace{2, 5, 3, 3, 6, 8, 2, 9, 5}_{\text{phase 1}}, \underbrace{7, 1, 2, 5, 2, 3, 7, 4}_{\text{phase 2}}, \underbrace{8, 1, 2, 7, 5, 3}_{\text{phase 3}}, \underbrace{6, 9, 6, 10, 4, 1, 2}_{\text{phase 4}} \ldots$$

**Fast Memory:**

$$\boxed{10}\boxed{6}\boxed{1}\boxed{9}\boxed{4}\boxed{2}$$

**Observations:**

- At the end of a phase, the fast memory entries are exactly the $k$ pages of that phase
- At the beginning of a phase, all entries get unmarked
- #page faults depends on #new pages in a phase