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Algorithm Theory
Sample Solution Exercise Sheet 5

Exercise 1: Matching & Vertex Cover in Bipartite Graphs *(5+5+2 Points)*

Let $G = (V, E)$ be a graph and assume that $M^* \subseteq E$ is a maximum matching and that $S^* \subseteq V$ is a minimum vertex cover (i.e., M^* is a largest possible matching and S^* a smallest possible vertex cover). We have seen in the lecture that for every graph G, it holds that $|M^*| \leq |S^*|$ because the edges in M^* have to be covered by disjoint nodes in S^*. In this exercise, we assume that G is a *bipartite graph* and our goal is to show that in this case, it always holds that $|M^*| = |S^*|$.

a) Recall that we can solve the maximum bipartite matching problem by reduction to maximum flow. Also recall that if we are given a maximum matching M^* (and thus a maximum flow of the corresponding flow network), we can find a minimum s-t cut by considering the residual graph. Describe how such a minimum cut looks like.

   Hint: Consider the set of all nodes which can be reached from an unmatched node on the left side via an alternating path.

b) Use the above description to show that any bipartite graph G has a vertex cover S^* of size $|M^*|$.

c) Show that the same thing is not true for general graphs by showing that for every $\varepsilon > 0$, there exists a graph $G = (V, E)$ for which $|S^*| \geq (2 - \varepsilon)|M^*|$.

   Hint: First try to find any graph for which $|S^*| > |M^*|$.

Sample Solution

a) Let $B = (U \cup V, E)$ be a bipartite graph with maximum matching M^*. In the corresponding flow network there is a source node s which is connected to all nodes in U and a target node t to which all nodes in V are connected. For $u \in U$ and $v \in V$, there is an edge from u to v iff u and v are adjacent in B. All edges have capacities 1. Let f be the maximum flow that corresponds to M^* and R the residual graph w.r.t. f. We know from the lecture that (A^*, B^*) is a minimum cut where A^* is defined as the set of nodes which can be reached from s by a path in R on which each edge has a positive capacity. For an $u \in U$, the edge (s, u) has positive residual capacity iff u is not matched. As there are no edges in B directing from V to U, we know that all edges from U to V are forward edges and all edges from V to U are backward edges. If (u, v) for an $v \in V$ has positive residual capacity, there is no flow through (u, v) and we know $\{u, v\} \notin M^*$. If (v, u') for an $u' \in U$ has positive residual capacity, there is flow through (u', v) and we know $\{u', v\} \in M^*$. Hence, A^* consists of all nodes which can be reached from an unmatched node on the left side via an alternating path.

b) Define $S^* = (U \cap B^*) \cup (V \cap A^*)$.

   S^* is a set cover: S^* covers all edges with left endpoint in B^* or right endpoint in A^*. To show that S^* is a vertex cover we need to show that there is no edge in the graph with left endpoint in A^* and right endpoint in B^*. Assume $e = \{u, v\}$ was such an edge. As $u \in A^*$, there is an alternating path to u. So if $e \notin M^*$, we could extend this path to v and therefore have $v \in A^*$, a contradiction. Otherwise, if $e \in M^*$, an alternating path reaching u must also contain v which implies that also $v \in A^*$, a contradiction.

$|S^*| \leq |M^*|$: We show that every node in S^* is an endpoint of an edge in M^* but never both endpoints of an edge in M^* are in S^*. Let $x \in S^*$. Every unmatched node in U is in A^*, i.e., all nodes in $U \cap B^*$ are matched. Let $v \in V \cap A^*$. If v was unmatched, the alternating path from some unmatched $u \in U$ to v would be an augmenting path and exchanging matched and unmatched edges would result in a larger matching which is a contradiction to M^* being maximum.

c) Let $\varepsilon > 0$. Choose n large enough such that $2/n \leq \varepsilon$ and consider K_n, the clique with n nodes. The size of a matching of any n-node graph is at most $n/2$. A vertex cover of K_n is of size at least $n-1$, because if two nodes u and v where not in the cover, then the edge between them would not be covered. It follows that for K_n we have

$$|S^*| \geq n - 1 = \left(2 - \frac{2}{n}\right)\frac{n}{2} \geq (2-\varepsilon)|M^*| \ .$$

Exercise 2: Matching in Regular Graphs *(5+5 Points)*

The degree of a node in a graph is the number of its neighbors. A graph is called r-regular for an $r \in \mathbb{N}$ if all nodes have degree r.

a) Show that any regular bipartite graph has a perfect matching.

b) Show that an n-regular graph with $2n$ nodes has a matching of size at least $n/2$.

Sample Solution

a) Let $B = (U \cup V, E)$ be r-regular. Let $U' \subseteq U$. We count the nodes in $N(U')$ in the following way: For each $u \in U'$ we write down its neighbors. The resulting sequence S has length $3|U'|$. Each $v \in N(U')$ has three neighbors in U and therefore appears at most three times in S. Thus, when we delete all multiple counts of elements in S, we obtain a sequence of length at least $|S|/3 = |U'|$ and these are the elements in $N(U')$. So we have $|N(U')| \geq |U'|$. In particular, we have $|V| \geq |N(U)| \geq |U|$. Exchanging the roles of U and V yields $|U| \geq |V|$ and thus $|U| = |V|$. With Hall's theorem it follows that B has a perfect matching.

b) An n-regular graph with $2n$ nodes has exactly n^2 edges. We compute a matching of size at least $n/2$ using the following greedy algorithm: Pick an arbitrary edge e, add it to the matching and remove it from the graph together with all edges incident to e. Repeat until no edge is left. In each iteration, we remove at most $2n-1$ edges, so we can repeat this procedure at least $\lceil n/2 \rceil$ times.

Exercise 3: Cover all Edges *(8 Points)*

You are given an undirected graph $G = (V, E)$, a capacity function $c : V \to \mathbb{N}$, and a subset $U \subseteq V$ of nodes. The goal is to cover every edge with the nodes in U, where every node $u \in U$ can cover up to $c(u)$ of its incident edges.
Formally, we are interested in the existence of an assignment of the edges to incident nodes in U such that each node u gets assigned at most $c(u)$ of its incident edges.

Devise an efficient algorithm to determine whether or not such an assignment exists for a given subset U and a given cost function c and state its runtime.

Sample Solution

We formulate the problem as a flow problem. We flow-network looks as follows: We have a source node s, a target node t, one node for each $u \in U$ and one node for each $e \in E$. We have the following edges:

- An edge from s to each $u \in U$ with capacity $c(u)$
- For any $e = \{u, v\} \in E$ an edge from u to e and one from v to e with capacity 1 each (or any capacity ≥ 1)
- An edge from each $e \in E$ to t with capacity 1

The problem is solvable iff there is a flow of size $|E|$ with integer flow values on each edge.

The network has integer capacities, the maximum flow is at most $|E|$ and the network has $O(|V|^3)$ edges. Therefore, there is a maximum flow with integer flow values that can be computed with Ford-Fulkerson in time $O(|V|^5)$. The problem is solvable iff this computed flow equals $|E|$.