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Theoretical Computer Science - Bridging Course
Winter Term 2019/2020
Exercise Sheet 1

In case you wish to get feedback, submit electronically by 12:15, Monday, October 28.

Exercise 1: Induction *(7 Points)*

Assume that there are n infinitely long straight lines lying on the 2-dimensional plane in such a way that no two lines are parallel, and no three lines intersect in a single point. Prove by induction that these lines divide the plane into $(n^2 + n + 2)/2$ regions.

Sample Solution

A single line divides the plane into two regions, where by $2 = (1^2 + 1 + 2)/2$, the base case is proved. Now, let us assume that n such lines divide the plane into $(n^2 + n + 2)/2$ regions (the induction hypothesis). Then, it is enough to show that $n + 1$ such lines divide the plane into $((n + 1)^2 + (n + 1) + 2)/2$.
Having $n+1$ lines lied on a plane as we assumed. Then, by ignoring an arbitrary line L, the remaining n lines must divide the plane into $(n^2 + n + 2)/2$ regions due to the induction hypothesis. Due to our assumptions, line L intersects each of the n lines in exactly one point. This way, L is divided into $n + 1$ segments, each segment divides a preexisting region into two regions. This leads to an increase in the number of regions by $n + 1$. Hence, the number of regions by $n + 1$ lines would be

$$\frac{(n^2 + n + 2)}{2} + (n + 1) = \frac{n^2 + n + 2 + 2n + 2}{2} = \frac{(n + 1)^2 + (n + 1) + 2}{2} ,$$

proving the induction step.

Exercise 2: Any Even Degree Node? *(5 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ of an undirected graph $G = (V, E)$ is the number of its neighbors, i.e,

$$d(v) = |\{u \in V \mid \{v, u\} \in E\}|.$$

Show that every simple graph with an odd number of nodes contains a node with even degree.

Sample Solution

Let $G = (V, E)$ be a simple graph, where $|V|$ is odd. Every edge contributes 2 to D, hence $D = 2|E|$. Therefore, D is an even number. For the sake of contradiction, let us assume that each and every node has an odd degree. Hence, since the sum of an odd number of odd numbers is also odd, D as the sum of all the nodes' degrees is an odd number, leading to a contradiction. Thus, there has to be a node with even degree.

Exercise 3: Counting Edges in Acyclic Graphs *(8 Points)*

A tree is an acyclic, connected, simple graph. Show that a tree with $n \geq 1$ nodes has $n-1$ edges. A forest is a (possibly unconnected) graph, where each connected component is a tree. Show that a forest consisting of k connected components has $n-k$ edges.

Sample Solution

First we show that an acyclic, connected graph with n nodes has exactly $n-1$ edges using an induction argument on n. A graph with just one node has $n-1=0$ edges. Assume that the statement holds for graphs with an arbitrary but fixed number of nodes n and consider a graph G with $n+1$ nodes. We remove one edge e, which makes G disintegrate into two components G_1 and G_2 which are not connected to each other (if there were a connection between G_1 and G_2, then reattaching e to G would close a cycle).
The components G_1 and G_2 themselves are still acyclic and (internally) connected and have $1 \leq k, m \leq n$ nodes with $k+m=n+1$. Using the induction hypothesis ($k, m \leq n$) we have that G_1 has $k-1$ edges and G_2 has $m-1$ edges. Since we removed exactly one edge to obtain G_1 and G_2, G has $(k-1)+(m-1)+1=n$ edges.
Next we show that a forest G consisting of k trees $G_1, \ldots, G_k$ has $n-k$ edges. Let n_i, $i \in \{1, \ldots, k\}$ the number of nodes of the i-th tree. Of course $\sum_{i=1}^{k} n_i = n$. We already know that G_i has $n_i - 1$ edges. Thus G has exactly $\sum_{i=1}^{k} n_i - 1 = n-k$ edges.