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Theoretical Computer Science - Bridging Course
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Exercise Sheet 9

for getting feedback submit electronically by 12:15, Monday, January 13, 2020

Exercise 1: Propositional Logic: Basic Terms *(2+2+2+2 Points)*

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \land \neg q) \lor (\neg p \lor q)$

(b) $\varphi_2 = (\neg p \land (\neg p \lor q)) \leftrightarrow (p \lor \neg q)$

(c) $\varphi_3 = (p \land \neg q) \to \neg(p \land q)$

(d) $\varphi_4 = (p \land q) \to (p \lor r)$

Remark: $a \to b :\equiv \neg a \lor b$, $a \leftrightarrow b :\equiv (a \to b) \land (b \to a)$, $a \not\to b :\equiv \neg(a \to b)$.

Sample Solution

(a) See Table 1. The result shows that φ_1 is a tautology.

(b) See Table 2. The result shows that φ_2 is satisfiable.

(c) φ_3 is equivalent to $\neg(p \land \neg q) \lor (\neg p \lor \neg q)$ which is equivalent to $(\neg p \lor q) \lor (\neg p \lor \neg q)$ which is equivalent to $\neg p \lor q \lor \neg p \lor \neg q$ which is equivalent to $\neg p \lor \neg q \lor q$ which is a tautology as either q or $\neg q$ holds.

(d) See Table 3. The result shows that φ_4 is tautology.

Exercise 2: CNF and DNF *(2+2 Points)*

(a) Convert $\varphi_1 := (p \to q) \to (\neg r \land q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \to \neg q) \land \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

model	p	q	$p \wedge \neg q$	$\neg p \vee q$	φ_1
✓	0	0	0	1	1
✓	0	1	0	1	1
✓	1	0	1	1	1
✗	1	1	0	1	1

Table 1: Truthtables for Exercises 1 (a).

model	p	q	$\neg p \vee q$	$\neg p \wedge (\neg p \vee q)$	$p \vee \neg q$	φ_2
✓	0	0	1	1	1	1
✗	0	1	1	1	0	0
✗	1	0	0	0	1	0
✗	1	1	1	0	1	0

Table 2: Truthtables for Exercises 1 (b).

model	p	q	r	$p \wedge q$	$p \vee r$	φ_4
✓	0	0	0	0	0	1
✓	0	0	1	0	1	1
✓	0	1	0	0	0	1
✓	0	1	1	0	1	1
✓	1	0	0	0	1	1
✓	1	0	1	0	1	1
✓	1	1	0	1	1	1
✓	1	1	1	1	1	1

Table 3: Truthtables for Exercises 1 (d).

Sample Solution

(a)

$$
\begin{aligned}
& (p \to q) \to (\neg r \wedge q) && \\
\equiv\ & \neg(\neg p \vee q) \vee (\neg r \wedge q) && \text{Definition of '}\to\text{'} \\
\equiv\ & (p \wedge \neg q) \vee (\neg r \wedge q) && \text{De Morgan} \\
\equiv\ & \big((p \wedge \neg q) \vee \neg r\big) \wedge \big((p \wedge \neg q) \vee q\big) && \text{Distribution} \\
\equiv\ & \big((p \vee \neg r) \wedge (\neg q \vee \neg r)\big) \wedge \big((p \vee q) \wedge (\neg q \vee q)\big) && \text{Distribution} \\
\equiv\ & \big((p \vee \neg r) \wedge (\neg q \vee \neg r)\big) \wedge \big((p \vee q) \wedge 1\big) && \text{Complementation} \\
\equiv\ & \big((p \vee \neg r) \wedge (\neg q \vee \neg r)\big) \wedge (p \vee q) && \text{Identity} \\
\equiv\ & (p \vee \neg r) \wedge (\neg q \vee \neg r) \wedge (p \vee q) && \text{Associativity}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \neg((\neg p \to \neg q) \wedge \neg r) && \\
\equiv\ & \neg((p \vee \neg q) \wedge \neg r) && \text{Definition of '}\to\text{'} \\
\equiv\ & \neg(p \vee \neg q) \vee r && \text{De Morgan} \\
\equiv\ & (\neg p \wedge q) \vee r && \text{De Morgan}
\end{aligned}
$$

Exercise 3: Logical Entailment *(2+2 Points)*

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \vee q, \neg r \vee p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b) $\varphi_2 := (q \leftrightarrow r) \to p$

Sample Solution

(a) KB does not entail φ_1. Consider the interpretation $I : p \mapsto 1, q \mapsto 0, r \mapsto 0$. Interpretation I is a model for KB but not for φ_1.

(b) Table 4 shows that every model of KB is also a model of φ_2, hence $KB \models \varphi_2$.

Exercise 4: Inference Rules and Calculi *(2+2 Points)*

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* $\mathbf{C}$ is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus $\mathbf{C}$ to 'generate' new formulae until ψ is obtained.

model of KB	p	q	r	$p \lor q$	$\neg r \lor p$	$q \leftrightarrow r$	φ_2	model of φ_2
✗	0	0	0	0	0	1	0	✗
✗	0	0	1	0	0	0	1	✓
✓	0	1	0	1	1	0	1	✓
✗	0	1	1	1	0	1	0	✗
✓	1	0	0	1	1	1	1	✓
✓	1	0	1	1	1	0	1	✓
✓	1	1	0	1	1	0	1	✓
✓	1	1	1	1	1	1	1	✓

Table 4: Truthtable for Exercise 3 (b).

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \land \psi}{\varphi, \psi}, \frac{(\varphi \land \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \land q, p \to r, (q \land r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Sample Solution

(a) We use $\mathbf{C_1}$ to derive new formulae until we obtain the desired one.

$$\begin{aligned}
\neg q \to r &\overset{\text{2nd rule}}{\vdash_{\mathbf{C_1}}} \neg r \to q \\
p \leftrightarrow \neg r &\overset{\text{3rd rule}}{\vdash_{\mathbf{C_1}}} p \to \neg r, \neg r \to p \\
p \to \neg r, \neg r \to q &\overset{\text{1st rule}}{\vdash_{\mathbf{C_1}}} p \to q
\end{aligned}$$

(b) We use $\mathbf{C_2}$ to derive new formulae until we obtain the desired one.

$$\begin{aligned}
p \land q &\overset{\text{2nd rule}}{\vdash_{\mathbf{C_2}}} p, q \\
p, p \to r &\overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} r \\
(q \land r) \to s &\overset{\text{3rd rule}}{\vdash_{\mathbf{C_2}}} q \to (r \to s) \\
q, q \to (r \to s) &\overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} r \to s \\
r, r \to s &\overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} s
\end{aligned}$$