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Theoretical Computer Science - Bridging Course
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Exercise Sheet 10

for getting feedback submit electronically by 12:15, Monday, January 20, 2020

Exercise 1: Resolution Calculus *(3+3 Points)*

Considering each of the following cases, first convert the knowledge base (KB_i) and the formula (φ_i) to CNFs. Then, by resolution, show that the knowledge base entails the formula.

(a) $KB_1 := \{(x \wedge y) \rightarrow (z \vee w), \;\; y \rightarrow x, \;\; (z \wedge y) \rightarrow 0, \;\; y\}$
$\varphi_1 := w \wedge y$

(b) $KB_2 := \{\neg A \rightarrow B, \;\; B \rightarrow A, \;\; A \rightarrow (C \wedge D)\}$
$\varphi_2 := A \wedge C \wedge D$

Sample Solution

(a) $KB_1 = \{(\neg x \vee \neg y \vee z \vee w), (\neg y \vee x), (\neg z \vee \neg y), y\}$
$\varphi_1 = w \wedge y$
We add the negation of the formula to the knowledge base and try to reach a contradiction by applying the resolution rule.

$KB_1' := \{(\neg x \vee \neg y \vee z \vee w), (\neg y \vee x), (\neg z \vee \neg y), y, (\neg w \vee \neg y)\}$

$$\{(\neg x \vee \neg y \vee z \vee w),\ y\} \vdash_{\mathbf{R}} \{(\neg x \vee z \vee w)\}$$
$$\{(\neg y \vee x),\ y\} \vdash_{\mathbf{R}} \{x\}$$
$$\{(\neg x \vee z \vee w),\ x\} \vdash_{\mathbf{R}} \{(z \vee w)\}$$
$$\{(\neg z \vee \neg y),\ y\} \vdash_{\mathbf{R}} \{\neg z\}$$
$$\{(z \vee w),\ \neg z\} \vdash_{\mathbf{R}} \{w\}$$
$$\{(\neg w \vee \neg y),\ y\} \vdash_{\mathbf{R}} \{\neg w\}$$
$$\{w,\ \neg w\} \vdash_{\mathbf{R}} []$$

(b) $KB_2 = \{(A \vee B), (\neg B \vee A), (\neg A \vee C), (\neg A \vee D)\}$
$\varphi_2 = A \wedge C \wedge D$
We add the negation of the formula to the knowledge base and try to reach a contradiction by applying the resolution rule.

$$KB_2' := \{(A \vee B), (\neg B \vee A), (\neg A \vee C), (\neg A \vee D), (\neg A \vee \neg C \vee \neg D)\}$$

$$\begin{aligned}
\{(A \vee B), (\neg B \vee A)\} &\vdash_{\mathbf{R}} \{A\} \\
\{(\neg A \vee C), A\} &\vdash_{\mathbf{R}} \{C\} \\
\{(\neg A \vee D), A\} &\vdash_{\mathbf{R}} \{D\} \\
\{(\neg A \vee \neg C \vee \neg D), A\} &\vdash_{\mathbf{R}} \{(\neg C \vee \neg D)\} \\
\{(\neg C \vee \neg D), C\} &\vdash_{\mathbf{R}} \{\neg D\} \\
\{\neg D, D\} &\vdash_{\mathbf{R}} []
\end{aligned}$$

Exercise 2: Implication vs. Entailment *(5 Points)*

Show that $P \models Q \leftrightarrow (True \models P \rightarrow Q)$

Sample Solution

Let $T(P)$ and $T(Q)$ be the set of models for for P and Q respectively.

(a) ($\rightarrow$): Let us assume that $P \models Q$. By the definition of entailment, we have $T(P) \subseteq T(Q)$. Moreover, since there is no interpretation under which Q and $\neg Q$ are both true, $T(Q) \cap T(\neg Q) = \emptyset$. Therefore, it implies that $T(P) \cap T(\neg Q) = \emptyset$. It means that there is no interpretation under which P is true while Q is false. Hence, under all interpretations, $P \rightarrow Q$ is true. That is $T(P \rightarrow Q) = T(True)$, and consequently $T(True) \subseteq T(P \rightarrow Q)$. By the definition of entailment, it concludes $True \models P \rightarrow Q$.

(b) ($\leftarrow$): Let us assume that $True \models P \rightarrow Q$. By the definition of entailment, this means that $P \rightarrow Q$ is true under all interpretations, and therefore there is no interpretation under which P is true and Q is false, i.e., $T(P) \cap T(\neg Q) = \emptyset$. Therefore, $T(P) \subseteq T(Q)$ and we can conclude that $P \models Q$.

(a) and (b) together concludes the prove of the statement.

Exercise 3: Understanding First Order Logic *(2+2+2 Points)*

Consider the following **first order logical** formulae

$$\begin{aligned}
\varphi_1 &:= \forall x R(x, x) \\
\varphi_2 &:= \forall x \forall y\, R(x, y) \rightarrow (\exists z R(x, z) \wedge R(z, y)) \\
\varphi_3 &:= \exists x \exists y\, (\neg R(x, y) \wedge \neg R(y, x))
\end{aligned}$$

where x, y are variable symbols and R is a binary predicate. Give an interpretation

(a) I_1 which is a **model** of $\varphi_1 \wedge \varphi_2$.

(b) I_2 which is **no model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

(c) I_3 which is a **model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

Sample Solution

(a) Pick $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$ where $R^{I_1}(x, y) :\Longleftrightarrow x \leq_{\mathbb{R}} y$.

This is a model because '$\leq_{\mathbb{R}}$' is *reflexive*, therefore fulfills φ_1. Moreover for every $x, y \in \mathbb{R}$ with $x \leq_{\mathbb{R}} y$ we can choose $z := x$, which fulfills $x \leq_{\mathbb{R}} z \wedge z \leq_{\mathbb{R}} y$. Thus φ_2 is also satisfied.

(b) Pick $I_2 := \langle \mathbb{R}, \cdot^{I} \rangle$ where $R^{I_2}(x, y) = \texttt{false}$.

This is not a model since it violates φ_1, e.g. $R^{I_2}(5, 5) = \texttt{false}$.

(c) Take two disjoint copies of $\mathbb{R}$ and the standard $\leq_{\mathbb{R}}$ relation on each of them; if x and y are from different copies they are not related in $\mathbb{R}$. Formally let

$$I_3 := \langle \{(a,1) \mid a \in \mathbb{R}\} \dot{\cup} \{(a,2) \mid a \in \mathbb{R}\}, \cdot^{I_3} \rangle$$

where $R^{I_3}((a,g),(b,h)) \Leftrightarrow (g = h \text{ and } a \leq_{\mathbb{R}} b)$.

This is a model because $\leq_{\mathbb{R}}$ is *reflexive*, therefore I_3 fulfills φ_1. Furthermore for every two $x = (a,g)$ and $y = (b,h)$ with $R^{I_3}((a,g),(b,h))$, i.e., $g = h$, we can choose $z := (a,g)$ which fulfills $R^{I_3}((a,g),(a,g)) \wedge R^{I_3}((a,g),(b,h))$. Thus φ_2 is also satisfied. φ_3 is also satisfied, e.g., $(5,1)$ and $(7,2)$ are incomparable, i.e., we have neither $R^{I_3}((5,1),(7,2))$ nor $R^{I_3}((7,2),(5,1))$

Exercise 4: Truth Value *(1+1+1 Points)*

Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain (or universe) for the variables consists of:

(a) the positive real numbers,

(b) the integers,

(c) the nonzero real numbers.

Sample Solution

(a) This is false, since no matter how small a positive number x we might choose, if we assume $y = \sqrt{x/2}$, then $x = 2y^2$, and it will not be true that $x \leq y^2$.

(b) This is true, because we can take $x = -1$ as an example.

(c) This is true, since we take $x = -1$.