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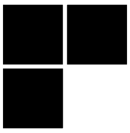
Algorithms Theory
Exercise Sheet 2

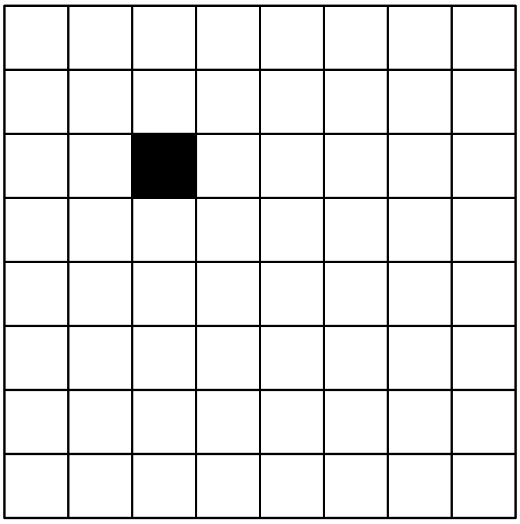
Due: Tuesday, 17th of November 2020, 4 pm

Exercise 1: Divide and Conquer *(10 Points)*

Consider a board with $n \times n$ cells with $n = 2^k$ for a $k \in \mathbb{N}_{\geq 1}$ (see below for an example). We have an unlimited supply of a specifically shaped tile, which covers exactly 3 cells of the board as follows:



The goal is to cover the board with tiles (which can be turned by 90, 180 and 270 degrees). We call an arrangement of tiles on the board a *valid tiling*, if all cells can be covered with the tile above *without any overlaps* and *without going over the edges* of the board. Assume that the input board has an arbitrary *single* cell that is initially covered (before the start of the algorithm). E.g. for $n = 8$ the board may look like this:



(a) Is there a *valid tiling* for every $2^k \times 2^k$ board ($k \in \mathbb{N}_{\geq 1}$) that is initially completely empty? Prove or disprove. *(2 Points)*

(b) Describe a *divide and conquer* algorithm that computes a *valid tiling* on a $n \times n$ board in $O(n^2)$ (with $n = 2^k, k \in \mathbb{N}_{\geq 1}$) that has *one* tiled cell. Assume that placing a tile is in $O(1)$. *(6 Points)*

(c) Show the running time of $O(n^2)$. *(2 Points)*

Exercise 2: Fast Fourier Transformation (FFT) *(10 Points)*

Let $p(x) = 7x^7 + 6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x$. We want to compute the discrete fourier transform $DFT_8(p)$ (where we define $DFT_8(p) := DFT_8(a)$ given that a is the vector of coefficients of p). More specifically, we want you to visualize the steps which the FFT-algorithm performs as follows.

(a) Illustrate the *divide* procedure of the algorithm. More precisely, for the i-th divide step, write down all the polynomials p_{ij} for $j \in \{0, \ldots, 2^i - 1\}$ that you obtain from further dividing the polynomials from the previous divide step $i-1$ (we define $p_{00} := p$). *(3 Points)*

(b) Illustrate the *combine* procedure of the algorithm. That is, starting with the polynomials of smallest degree as base cases, compute the $DFT_N(p_{ij})$ bottom up with the recursive formula given in the lecture (where N is the smallest power of 2 such that $\deg(p_{ij}) < N$). *(7 Points)*

Hints: The base case for a polynomial $p = a$ of degree 0 is $DFT_1(p) = DFT_1(a) = a$. In general, it also suffices to give the $p_{ij}(\omega)$ for the appropriate roots of unity ω, from which $DFT_N(p_{ij})$ can be derived. Use $\sqrt{\cdot}$ instead of floating point numbers if possible.