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Algorithms Theory
Exercise Sheet 8

Due: Tuesday, 12th of January 2020, 4 pm

Exercise 1: Property of Augmenting Paths *(10+5* Points)*

Let $G = (V, E)$ be a *weighted, directed* graph with $n = |V|$ nodes and $m = |E|$ edges, and let $s, t \in V$.

(a) Let f be a valid s-t-flow in G and assume that $m_f \leq m$ is the number of edges $e \in E$ for which $f(e) > 0$. Show that f can be "decomposed" into flows along at most m_f s-t paths P_i (i.e., $1 \leq i \leq m_f$). More precisely, show that there exist at most m_f paths P_i each with a flow value f_i, such that for each $e \in E$ we have $\sum_{i:e\in P_i} f_i \leq f(e)$ and such that $\sum_i f_i = |f|$. *(5 Points)*

Hint: You can give a constructive proof using a greedy algorithm similar to the one in the lecture.

(b) The residual graph G_f can also be interpreted as a flow network with source s and sink t. Show that the minimum s-t-cut in the residual graph G_f has capacity exactly $|f^*| - |f|$, where f^* is a maximum flow of G. *(5 Points)*

(c) Let $|f| < |f^*|$. Show that there is always an augmenting path P in G with respect to a given flow f such that $\text{bottleneck}(P, f) \geq (|f^*| - |f|)/m$. *(5 Points)*

Exercise 2: Vertex Connectivity Refined *(10+5* Points)*

In the lecture we saw that a *minimum vertex cut* (MVC) can be computed in time $O(m \cdot n^3)$. Given that some parameters of the *undirected, unweighted* input graph $G = (V, E)$ are fairly small, we want to improve that runtime.

(a) Let κ be the vertex connectivity of G, i.e., the number of vertices in an MVC. Show that in order to compute a *global* MVC, it suffices to compute minimum s-t vertex cuts for $O(n \cdot \kappa)$ different source-target pairs (s, t). *(3 Points)*

Hint: Show that picking an arbitrary set of $\kappa + 1$ sources suffices.

(b) Give an algorithm to find an MVC in time $O(m \cdot n \cdot \kappa^2)$. *(2 Points)*

Remark: You can assume for now that κ is already known. If you did not succeed in part (a), you might still be able to show a running time of $O(m \cdot n^2 \cdot \kappa)$ here.

(c) How can the same asymptotic running time be achieved without knowing κ in advance? *(3 Points)*

(d) Let δ be the minimum degree of G (i.e., the minimum number of incident edges of some node in G). Show that it even suffices to compute minimum s-t vertex cuts for $O(n + \delta^2)$ pairs $s, t \in V$. Argue why this is the case and how you can find these $O(n + \delta^2)$ s,t-pairs. *(6 Points)*

Hint: Let v be some node of degree δ and make a case distinction depending on if there exists an MVC that does not contain v or if v is contained in all MVCs.

(e) Explain how you get an overall running time of $O\big(m\kappa \cdot \min(n+\delta^2, \kappa n)\big)$ for computing an MVC in a graph G with vertex connectivity κ and minimum degree δ. *(1 Point)*

**10 points are bonus. They do not increase the number of required points for the "Studienleistung".*