Exercise 1: Matching vs Vertex Cover  

Given an undirected Graph $G = (V, E)$, a vertex cover of $G$ is a set of nodes $S \subseteq V$ such that for all $\{u, v\} \in E$, we have $\{u, v\} \cap S \neq \emptyset$. A minimum vertex cover is a vertex cover of minimum size.

a) Show that for a maximum matching $M^*$ and a minimum vertex cover $S^*$ we have $|M^*| \leq |S^*|$.  
(2 Points)

Next we want to show that in bipartite graphs, it also holds $|S^*| \leq |M^*|$.

b) Recall that we can solve the maximum bipartite matching problem by reduction to maximum flow. Also recall that if we are given a maximum matching $M^*$ (and thus a maximum flow of the corresponding flow network), we can find a minimum $s$-$t$ cut by considering the residual graph. Describe how such a minimum cut looks like.

*Hint: Consider the set of all nodes which can be reached from an unmatched node on the left side via an alternating path.*  
(3 Points)

c) Use the above description to show that any bipartite graph $G$ has a vertex cover $S^*$ of size $|M^*|$.  
(3 Points)

d) Show that the same thing is not true for general graphs by showing that for every $\varepsilon > 0$, there exists a graph $G = (V, E)$ for which $|S^*| \geq (2 - \varepsilon)|M^*|$.  

*Hint: First try to find any graph for which $|S^*| > |M^*|$.  
(2 Points)

Exercise 2: Contention Resolution  

Show that for the randomized algorithm for contention resolution from the lecture, the expected time until all processes have been successful is $O(n \log n)$.  

(10 Points)