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Algorithms Theory
Exercise Sheet 10

Due: Tuesday, 26th of January 2021, 4 pm

Exercise 1: Max Cut *(10 Points)*

Let $G = (V, E)$ with $n = |V|, m = |E|$ be an undirected, unweighted graph. Consider the following randomized algorithm: Every node $v \in V$ joins the set S with probability $\frac{1}{2}$. The output is $(S, V \setminus S)$.

(a) What is the probability to obtain a cut? *(1 Point)*

(b) For $e \in E$ let random variable $X_e = 1$ if e crosses the cut, and $X_e = 0$, else. Let $X = \sum_{e \in E} X_e$. Compute the expectation $\mathbb{E}[X]$ of X. *(2 Points)*

(c) Show that with probability at least $1/3$ this algorithm outputs a cut of size at least $m/4$. (that is a cut of maximum possible size can be at most 4 times as large). *(4 Points)*

Remark: For a non-negative random variable X, the Markov inequality states that for all $t > 0$ we have $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$.

(d) Show how to use the above algorithm to obtain a cut with at least $m/4$ edges with probability at least $1 - \left(\frac{2}{3}\right)^k$ for $k \in \mathbb{N}$. *(3 Points)*

Remark: If you did not succeed in (c) you can use the result as a black box for (d).

Exercise 2: Ternary Tree *(10 Points)*

Consider a full, complete ternary tree where each *inner* node has exactly three child nodes. Note that since the tree is full and complete, all leaves have the same distance (= height) ℓ from the root. Let n be the number of leaves of the tree.
Each leave is given a boolean value. The value of an inner node is defined recursively as the majority value of its three direct children. The objective is to compute the value of the root. The performance of an algorithm to solve this problem is measured by the number of values of leaves it reads.

(a) Is there a *deterministic* algorithm to determine the value of the root, such that for any given input, the algorithm does not need to read the values of *all* leaves? Explain your answer carefully. *(3 Points)*

(b) Give a *recursive, randomized* algorithm (analysis in part (c)) that *always* determines the value of the root but reads at most a^ℓ leaves *in expectation* for $a < 3$. *(4 Points)*

Remark: You can also show that only a q^ℓ-fraction of all leaves is read in expectation for $q < 1$.

(c) Based on the algorithm of (b) give a *tight* upper bound (as a function of the number of leaves n) for the *expected* number of leaves that are read by the algorithm. *(3 Points)*