Algorithm Theory

Chapter 7
Randomized Algorithms

Part VIII:
Cut Counting and Edge Sampling

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Number of Minimum Cuts

- Given a graph G, how many minimum cuts can there be?

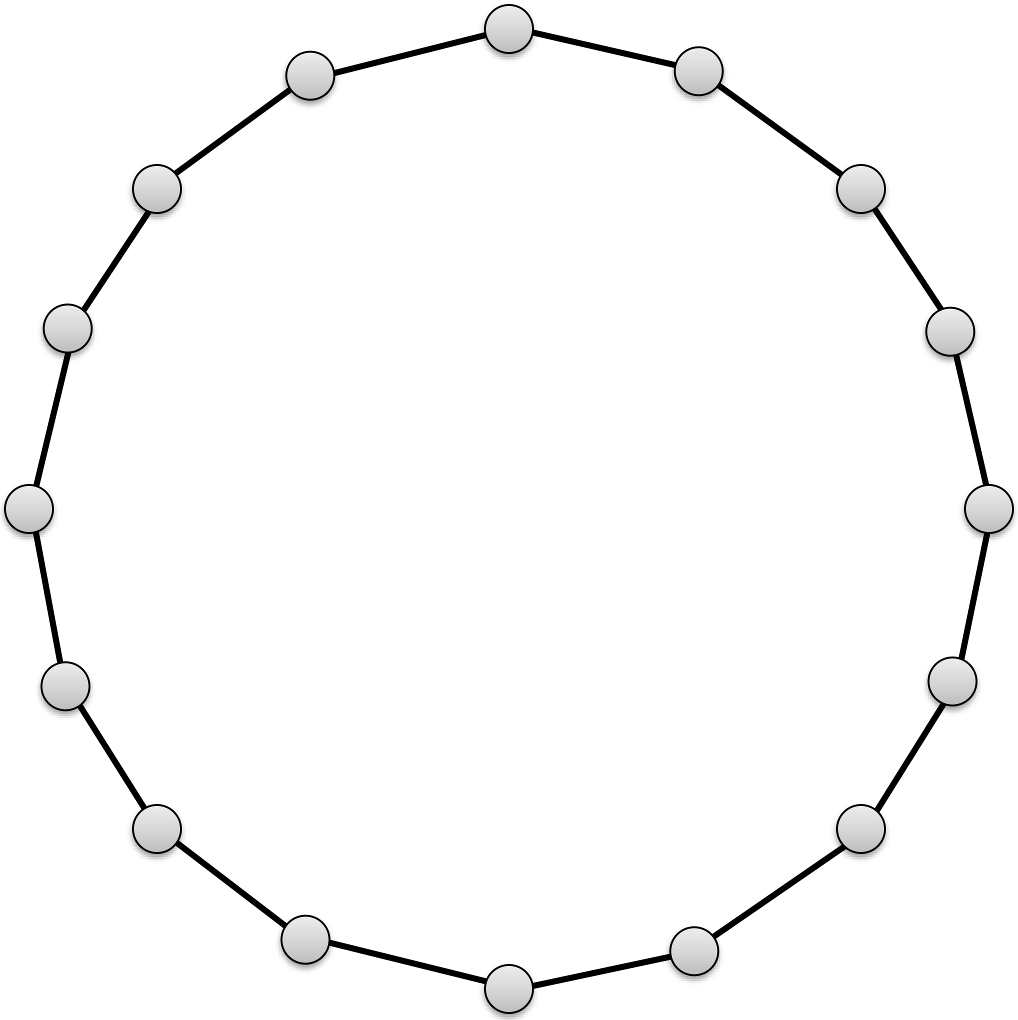
- Or alternatively: If G has edge connectivity λ, how many ways are there to remove λ edges to disconnect G?

- Note that the total number of cuts is large:

$$\#\text{cuts} = \frac{2^n - 2}{2} = 2^{n-1} - 1$$

Number of Minimum Cuts

Example: Ring with n nodes



- Minimum cut size: 2
- Every two edges induce a min cut
- Number of edge pairs:

$$\binom{n}{2}$$

- Are there graphs with more min cuts?

Number of Min Cuts

Theorem: The number of minimum cuts of a connected graph is at most $\binom{n}{2}$.

Proof:

- Assume there are s min cuts
- For $i \in \{1, \dots, s\}$, define event $\mathcal{C}_i$:

  $\mathcal{C}_i \coloneqq \{\text{basic contraction algorithm returns min cut } i\}$

- We know that for $i \in \{1, \dots, s\}$: $\mathbb{P}(\mathcal{C}_i) \geq 1/\binom{n}{2}$
- Events $\mathcal{C}_1, \dots, \mathcal{C}_s$ are disjoint:

$$\mathbf{1} \geq \mathbb{P}\left(\bigcup_{i=1}^{s} \mathcal{C}_i\right) = \sum_{i=1}^{s} \mathbb{P}(\mathcal{C}_i) \geq \frac{s}{\binom{n}{2}} \quad \Rightarrow \quad s \leq \binom{n}{2}$$

Counting Larger Cuts

- In the following, assume that min cut has size $\lambda = \lambda(G) \geq 1$
- How many cuts of size $\leq k = \alpha \cdot \lambda$ can a graph have?
- Consider a specific cut (A, B) of size $\leq k$
- As before, during the contraction algorithm:
  - min cut size $\geq \lambda$
  - total number of edges $\geq \lambda \cdot \#\text{nodes}/2$
  - cut (A, B) remains as long as none of its edges gets contracted
- Prob. that (A, B) survives i^{th} contraction (if it still exists)

$$= 1 - \frac{k}{\#\text{edges}} \geq 1 - \frac{2\alpha\lambda}{\lambda \cdot \#\text{nodes}} = 1 - \frac{2\alpha}{n - i + 1} = \frac{n - 2\alpha - i + 1}{n - i + 1}$$

For simplicity, in the following, assume that 2α is an integer

Counting Larger Cuts

Lemma: If $2\alpha \in \mathbb{N}$, the probability that cut (A, B) of size $\leq \alpha \cdot \lambda$ survives the first $n - 2\alpha$ edge contractions is at least

$$\frac{(2\alpha)!}{n(n-1) \cdot \ldots \cdot (n - 2\alpha + 1)} \geq \frac{2^{2\alpha-1}}{n^{2\alpha}}.$$

Proof:

- As before, event $\mathcal{E}_i$: cut (A, B) survives contraction i

$$\mathbb{P}\left(\bigcap_{i=1}^{n-2\alpha} \mathcal{E}_i\right) = \prod_{i=1}^{n-2\alpha} \mathbb{P}(\mathcal{E}_i | \mathcal{E}_1 \cap \dots \cap \mathcal{E}_{i-1}) \geq \prod_{i=1}^{n-2\alpha} \frac{n - 2\alpha - i + 1}{n - i + 1}$$

$$= \frac{n-2\alpha}{n} \cdot \frac{n-2\alpha-1}{n-1} \cdot \frac{n-2\alpha-2}{n-2} \cdot \dots \cdot \frac{2}{2\alpha+2} \cdot \frac{1}{2\alpha+1}$$

$$= \frac{2\alpha \cdot (2\alpha-1) \cdot \dots \cdot 1}{n \cdot (n-1) \cdot \dots \cdot (n-2\alpha+1)} \geq \frac{(2\alpha)!}{n^{2\alpha}} \geq \frac{2^{2\alpha-1}}{n^{2\alpha}}$$

Number of Cuts

Theorem: If $2\alpha \in \mathbb{N}$, the number of edge cuts of size at most $\alpha \cdot \lambda(G)$ of a connected n-node graph G is at most $n^{2\alpha}$.

Proof:

$$\mathbb{P}(\text{cut of size} \le \alpha \cdot \lambda \text{ survives } \underbrace{\text{first } n - 2\alpha \text{ contractions}}_{\text{afterwards: } 2\alpha \text{ nodes}}) \ge \frac{2^{2\alpha-1}}{n^{2\alpha}}$$

2α nodes $< 2^{2\alpha-1}$ cuts ⇨ return random remaining cut

We get a randomized algorithm that returns any specific cut (A, B) of size $\le \alpha \cdot \lambda$ with probability at least $1/n^{2\alpha}$.

$\Longrightarrow$ Now the argument is the same as for cuts of size λ.

Remark: The bound also holds in general, even if $2\alpha \notin \mathbb{N}$.

Resilience To Edge Failures

- Consider a network (a graph) G with n nodes and edge connectivity λ
- Assume that each link (edge) of G fails independently with probability p
- How large can p be such that the remaining graph is still connected with high probability or with probability $1 - \varepsilon$?

Maintaining Connectivity

Claim: A graph $G = (V, E)$ is connected if and only if every cut (A, B) has size at least 1.

Proof:

- If there is a cut (A, B) of size 0, there are no edges between the nodes in A and B and G is therefore not connected.

- Now, assume that G is not connected
  - G consists of at least 2 different connected components
  - Let A be the set of nodes of one connected component
    $\Longrightarrow (A, V \setminus A)$ is a cut of size 0

For G to remain connected, we need to make sure that ≥ 1 edge of every cut remains.

Resilience to Edge Failures

- Consider an edge cut (A, B) of size $k = \alpha \cdot \lambda(G)$
- Assume that each edge fails with probability $p \leq 1 - \frac{c \cdot \ln n}{\lambda(G)}$
- Hence each edge survives with probability $q \geq \frac{c \cdot \ln n}{\lambda(G)}$
- Probability that no edge crossing (A, B) survives

$$\mathbb{P}(\text{no edge of } (A,B) \text{ survives}) = p^k \leq \left(1 - \frac{c \cdot \ln n}{\lambda(G)}\right)^{\alpha \cdot \lambda(G)}$$

$$\leq e^{-c\alpha \ln n} = \frac{1}{n^{\alpha \cdot c}}$$

$\forall x \in \mathbb{R} : 1 + x \leq e^x$

Maintaining All Cuts of a Certain Size

Claim: If each edge survives with prob. $q \geq \frac{c \cdot \ln n}{\lambda(G)}$, for a given $k = \alpha\lambda(G)$, at least one edge of each cut of size exactly k survives with prob. at least

$$1 - \frac{1}{n^{(c-2)\alpha}}.$$

- Number the cuts of size k from 1 to $s \leq n^{2\alpha}$
- Event $\mathcal{F}_i$: all edges of i^{th} cut of size k are removed

From before: $\forall i \in \{1, \dots, s\} : \mathbb{P}(\mathcal{F}_i) \leq \frac{1}{n^{\alpha c}}$

$$\Longrightarrow \quad \mathbb{P}\left(\bigcup_{i=1}^{s} \mathcal{F}_i\right) \leq \sum_{i=1}^{s} \mathbb{P}(\mathcal{F}_i) \leq \frac{n^{2\alpha}}{n^{\alpha c}} = \frac{1}{n^{(c-2)\alpha}}$$

Maintaining Connectivity

Theorem: If each edge of a (simple) n-node graph G independently fails with probability at most $1 - \frac{(c+4) \cdot \ln n}{\lambda(G)}$, the remaining graph is connected with probability at least $1 - \frac{1}{n^c}$.

Proof:

- $\underbrace{\mathbb{P}(\exists \text{ cut of size } k = \alpha\lambda \text{ that loses all edges})}_{= P_k} \leq \frac{1}{n^{(c+2)\alpha}} \leq \frac{1}{n^{c+2}}$.
- #difference cut sizes $< n^2$ (max. possible cut size $= n^2/4$)
- **Union bound over all possible k:**

$$\mathbb{P}(\exists \text{ cut that loses all edges}) \leq \sum_{k=\lambda}^{n^2/4} P_k \leq n^2 \cdot \frac{1}{n^{c+2}} = \frac{1}{n^c}$$