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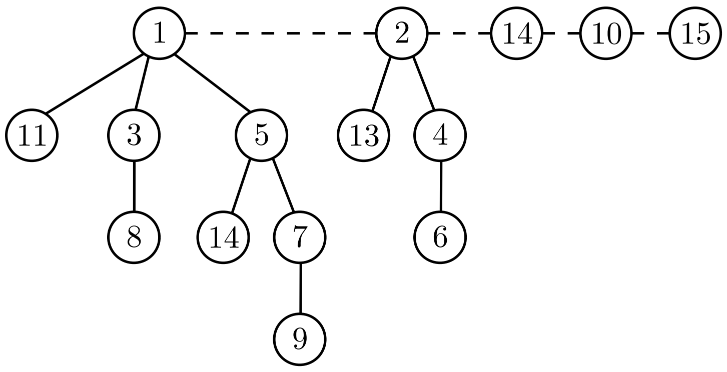
Algorithms Theory
Sample Solution Exercise Sheet 6

Due: Tuesday, 15th of December 2020, 4 pm

Exercise 1: Fibonacci Heap Worst Case *(7 Points)*

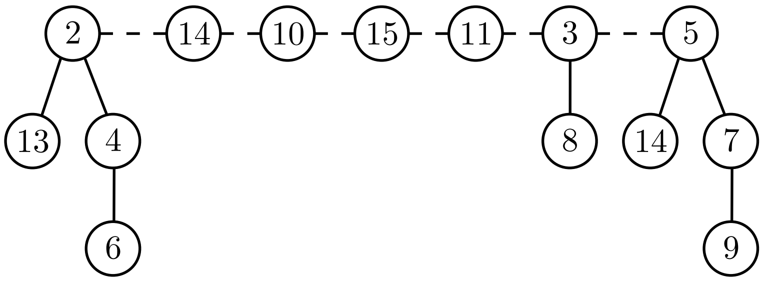
(a) Consider the Fibonacci heap given below. Perform a `delete-min` operation. Give the state *before and after* the `consolidate` operation. Conduct the `link` operations *exactly* in the order in which the algorithm given in the lecture does it (Chapter 5 Part IV Slide 16). *(4 Points)*

(b) Give a valid instance of a Fibonacci heap where `delete-min` has a *worst case* runtime of $\Omega(n)$ and explain why this is the case for that instance. *(3 Points)*

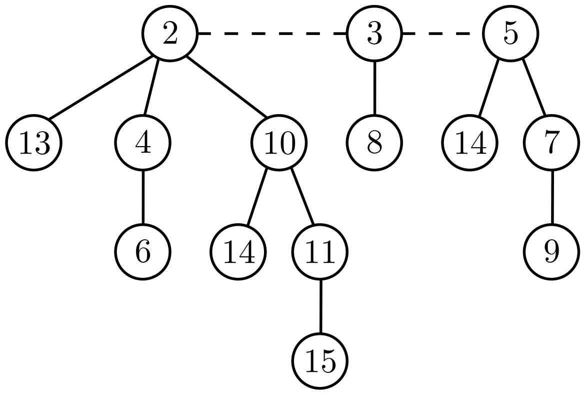


Sample Solution

(a) After `delete-min`, before `consolidate`



After `consolidate`



(b) Consider inserting the keys $1, \ldots, n$ into an empty Fibonacci heap. These are going to be inserted, in a lazy fashion, as heaps of rank zero into the rootlist. Then we run `delete-min`, which removes the heap with key 1 and triggers a `consolidate`. The `consolidate` routine iterates over the root-list, which still contains $n-1$ heaps. Therefore the total runtime of `consolidate` and consequentially of `delete-min` is $\Omega(n)$.

Exercise 2: Fibonacci Heap Modifications *(13 Points)*

(a) Assume that operation `decrease-key` never occurs. Show that in this case, the maximum rank $D(n)$ of a Fibonacci heap is at most $\lfloor \log_2(n) \rfloor$. *(5 Points)*

(b) We want to augment the Fibonacci heap data structure by adding an operation `increase-key`(v, k) to increase the key of a node v (given by a direct pointer) to the value k. The operation should have an amortized running time of $O(\log n)$. Describe the operation `increase-key`(v, k) in sufficient detail and prove the correctness and amortized running time. *(8 Points)*

Remark: You can use the same potential function as for the standard Fibonacci heap data structure. Note however that after conducting `increase-key`(v, k) *the Fibonacci heap must still be a list of heaps, where the maximum rank $D(n) \in O(\log n)$.*

Sample Solution

(a) First we show inductively that when there are *no* `decrease-key` operations, then a heap of rank i in the rootlist has exactly 2^i nodes (we call the number of nodes the size of the heap in the following). A heap of rank 0 is just a single node, thus it has size $2^0 = 1$.

Given a heap h of rank $i > 0$ which might also be a sub-heap attached to some parent node. The only way the degree i of h can be created is by linking two heaps h_1, h_2 of rank $i-1$. By induction hypothesis heaps of rank $i-1$ have 2^{i-1} elements. Therefore, the size of the heap $h = \texttt{link}(h_1, h_2)$ is the sum of the sizes of $h_, h_2$, i.e., $2^{i-1} + 2^{i-1} = 2^i$.

Remark: When we execute a `delete-min` *operation, then smaller heaps that are attached to the current minimum are cut and reinserted into the rootlist. But this does not change the the form of the subheaps of the root in any way, so the induction argument above remains valid.*

Since we have only n nodes in the Fibonacci heap in total, the heap with the biggest rank $D(n)$ must fulfill the inequality

$$2^{D(n)} \leq n \iff D(n) \leq \log_2 n.$$

Since $D(n)$ is an integer value we have $D(n) \leq \lfloor \log_2 n \rfloor$.

(b) **Implementation and correctness arguments**: As suggested in the remark, we try to design the `increase-key`(v, k) operation to maintain the same conditions of the Fibonacci heap. Specifically, we ensure in the following that each node looses at most one rank by loosing a child. First we assert that for `increase-key`(v, k) the new key k is larger than the current key of v. If k is smaller than or equal to all the keys of its child nodes, the heap condition is not violated after changing the key to k and we do nothing else.

Otherwise we first cut out and reinsert all child-heaps of v into the rootlist. Since v has lost too many children (each node can loose at most one) we also cut v from its parent and reinsert it as single node into the rootlist. Since v's former parent now lost a child, we run the cascading cut procedure on v's former parent, meaning that all successive marked ancestors of v are cut out and reinserted into the rootlist. The closest previously unmarked ancestor of v is marked.

Finally we have to consider a special case that forces us to do another step. If the node v whose key we increased is the current minimum, then we have to go through the whole rootlist to find the new minimum (or to confirm that v is still the minimum). But then we also have to run a consolidate like for `delete-min`. The reason for that is technical: we have to shrink the size of the rootlist R back down to $D(n)$ in order to "pay" that costly search (and consolidate) with the associated decrease in potential.

Runtime: The *actual cost* of our implementation of `increase-key`(v, k) is composed of the following components. We have $t_1 \leq D(n)$ steps for cutting and reinserting all child-heaps of v, since $D(n)$ is the maximum number of children v can have.

The next costly step is the cascading cuts procedure, which takes t_2, where t_2 is the number of successively marked ancestors of v plus one or, alternatively, the distance to the closest unmarked ancestor of v.

Finally, let us assume that we actually increase the key of current the minimum v. Then we have to find a new minimum and also consolidate, which takes time to the order of $t_3 \leq |H.\texttt{rootlist}|$, where R is the size of the rootlist.

The potential of the Fibonacci heap changes as follows:

$$\begin{aligned} R_{new} &= R_{old} + D(n) + 1 - |H.\texttt{rootlist}| \\ M_{new} &= M_{old} - (t_2 - 1) \\ \Phi_{new} &= \Phi_{old} + D(n) + 1 - |H.\texttt{rootlist}| - 2(t_2 - 1). \end{aligned}$$

The difference $\Phi_{new} - \Phi_{old}$ can be used to offset or more precisely *amortise* our true costs $t_1 + t_2 + t_3$:

$$\begin{aligned} a_i &= t_1 + t_2 + t_3 + \Phi_{new} - \Phi_{old} \\ &\leq D(n) + t_2 + |H.\texttt{rootlist}| + \Phi_{new} - \Phi_{old} \\ &= D(n) + t_2 + |H.\texttt{rootlist}| + D(n) + 1 - |H.\texttt{rootlist}| - 2(t_2 - 1) \\ &= 2D(n) + 3 - t_2 \\ &\leq 2D(n) + 3 \in \mathrm{O}(\log n). \end{aligned}$$