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Algorithms Theory
Sample Solution Exercise Sheet 9

Due: Tuesday, 19th of January 2021, 4 pm

Exercise 1: Matching vs Vertex Cover *(10 Points)*

Given an undirected Graph $G = (V, E)$, a *vertex cover* of G is a set of nodes $S \subseteq V$ such that for all $\{u, v\} \in E$, we have $\{u, v\} \cap S \neq \emptyset$. A minimum vertex cover is a vertex cover of minimum size.

a) Show that for a maximum matching M^* and a minimum vertex cover S^* we have $|M^*| \leq |S^*|$. *(2 Points)*

Next we want to show that in bipartite graphs, it also holds $|S^*| \leq |M^*|$.

b) Recall that we can solve the maximum bipartite matching problem by reduction to maximum flow. Also recall that if we are given a maximum matching M^* (and thus a maximum flow of the corresponding flow network), we can find a minimum s-t cut by considering the residual graph. Describe how such a minimum cut looks like.

   Hint: Consider the set of all nodes which can be reached from an unmatched node on the left side via an alternating path. *(3 Points)*

c) Use the above description to show that any bipartite graph G has a vertex cover S^* of size $|M^*|$. *(3 Points)*

d) Show that the same thing is not true for general graphs by showing that for every $\varepsilon > 0$, there exists a graph $G = (V, E)$ for which $|S^*| \geq (2 - \varepsilon)|M^*|$.

   Hint: First try to find any graph for which $|S^*| > |M^*|$. *(2 Points)*

Sample Solution

a) For each edge in the matching, at least one endpoint must be in the vertex cover. As a node can not cover more than one matching edge, there are at least as many nodes in the vertex cover as edges in the matching.

b) Let $B = (U \cup V, E)$ be a bipartite graph with maximum matching M^*. In the corresponding flow network there is a source node s which is connected to all nodes in U and a target node t to which all nodes in V are connected. For $u \in U$ and $v \in V$, there is an edge from u to v iff u and v are adjacent in B. All edges have capacities 1. Let f be the maximum flow that corresponds to M^* and R the residual graph w.r.t. f. We know from the lecture that (A^*, B^*) is a minimum cut where A^* is defined as the set of nodes which can be reached from s by a path in R on which each edge has a positive capacity. For an $u \in U$, the edge (s, u) has positive residual capacity iff u is not matched. As there are no edges in B directing from V to U, we know that all edges from U to V are forward edges and all edges from V to U are backward edges. If (u, v) for an $v \in V$ has positive residual capacity, there is no flow through (u, v) and we know $\{u, v\} \notin M^*$. If (v, u') for an $u' \in U$ has positive residual capacity, there is flow through (u', v) and we know $\{u', v\} \in M^*$. Hence, A^* consists of s and all nodes which can be reached from an unmatched node on the left side via an alternating path.

c) Define $S^* = (U \cap B^*) \cup (V \cap A^*)$.

S^* is a vertex cover: S^* covers all edges with left endpoint in B^* or right endpoint in A^*. To show that S^* is a vertex cover we need to show that there is no edge in the graph with left endpoint in A^* and right endpoint in B^*. Assume $e = \{u, v\}$ was such an edge. As $u \in A^*$, there is an alternating path to u. So if $e \notin M^*$, we could extend this path to v and therefore have $v \in A^*$, a contradiction. Otherwise, if $e \in M^*$, an alternating path reaching u must also contain v which implies that also $v \in A^*$, a contradiction.

$|S^*| = |M^*|$: We showed before that there is no edge between a node in $U \cap A^*$ and a node in $V \cap B^*$. As there is no edge directed from V to U this means that the edges going out of A^* are those from s to $U \cap B^*$ and from $V \cap A^*$ to t. These edges stand in a 1-1 correspondence to the nodes in S^*. So the size of the minimum cut (A^*, B^*) equals $|S^*|$. As the size of a min-cut also equals the maximum flow which equals $|M^*|$, we obtain $|S^*| = |M^*|$.

d) The statement even holds for $\varepsilon = 0$ and arbitrary large graphs. For an odd n, consider K_n, the clique with n nodes. The size of a matching of any n-node graph is at most $\lfloor n/2 \rfloor$ which equals $(n-1)/2$ if n is odd. A vertex cover of K_n is of size at least $n-1$, because if two nodes u and v were not in the cover, the edge between them would not be covered. So we have $|S^*| = 2|M^*|$.

Exercise 2: Contention Resolution *(10 Points)*

Show that for the randomized algorithm for contention resolution from the lecture, the expected time until all processes have been successful is $O(n \log n)$.

Sample Solution

Let T be the random variable that equals the time until all processes succeeded. The expected value of T is defined by

$$E[T] = \sum_{t=1}^{\infty} t \cdot \Pr(T = t) \ .$$

We define $t_i := (i+1) \cdot en \ln n$. We have

$$E[T] = \sum_{t=1}^{\infty} t \cdot \Pr(T = t) = \sum_{t=1}^{t_2} t \cdot \Pr(T = t) + \sum_{t=t_2+1}^{\infty} t \cdot \Pr(T = t)$$

We show

1. $\sum_{t=1}^{t_2} t \cdot \Pr(T = t) = O(n \log n)$
2. $\sum_{t=t_2+1}^{\infty} t \cdot \Pr(T = t) = O(1)$

1.

$$\sum_{t=1}^{t_2} t \cdot \Pr(T = t) \leq \sum_{t=1}^{t_2} t_2 \cdot \Pr(T = t) = t_2 \sum_{t=1}^{t_2} \Pr(T = t) \leq t_2 = 3en \ln n$$

2.

$$\sum_{t=t_2+1}^{\infty} t \cdot \Pr(T = t) = \sum_{i=2}^{\infty} \sum_{t=t_i+1}^{t_{i+1}} t \cdot \Pr(T = t) \leq \sum_{i=2}^{\infty} t_{i+1} \sum_{t=t_i+1}^{t_{i+1}} \Pr(T = t) \leq \sum_{i=2}^{\infty} t_{i+1} \Pr(T > t_i)$$

$$\overset{(*)}{\leq} \sum_{i=2}^{\infty} \frac{(i+2)(en \ln n)}{n^i} \leq \sum_{i=2}^{\infty} \frac{(i+2)en^2}{n^i} = \sum_{i=2}^{\infty} \frac{(i+2)e}{n^{i-2}} = \sum_{j=0}^{\infty} \frac{(j+4)e}{n^j}$$

$$= e \sum_{j=0}^{\infty} \frac{j}{n^j} + 4e \sum_{j=0}^{\infty} \frac{1}{n^j} = O(1)$$

At $(*)$ we used $\Pr(T > t_i) < \frac{1}{n^i}$ (known from the lecture).