Theoretical Computer Science - Bridging Course
Winter Term 2020/21
Exercise Sheet 1

In case you wish to get feedback, submit electronically by 12:15, Monday, November 9.

Exercise 1: Proof by Induction

Prove by induction for any positive integer number \( n \), \( n^3 + 2n \) is divisible by 3.

Exercise 2: Which Statement is True?

Let \( A, B \) and \( C \) be sets in some universal set \( U \). Which of the following statements is always true. Justify.

1. If \( A \cap B = A \cap C \), then \( B = C \).
2. If \( A \cup B = A \cup C \), then \( B = C \).
3. \( \overline{A \cup B} = \overline{A} \cap \overline{B} \). Remark: \( \overline{A} \) is the compliment of \( A \)

Exercise 3: Visiting All Nodes (Part 1)

A simple graph is a graph without self loops, i.e. every edge of the graph is an edge between two distinct nodes. A complete graph is a simple undirected graph in which every pair of distinct nodes is connected by a unique edge e.g. a triangle on 3 nodes.

Prove that every complete graph \( G \) has a path \( P \) that visits all the nodes of \( G \).

Exercise 4: Visiting All Nodes (Part 2)

A directed path \( P \) on \( n \) vertices is a simple directed graph whose edge set is the following set of ordered pairs \( \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1 \text{ and } v_i \text{ is a node in } P\} \) i.e. a path in which all the arrows point in the same direction as its steps. We write \( P = v_1v_2...v_n \) to denote the directed path \( P \).

A tournament is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a directed edge with any one of the two possible orientations.

Prove that every tournament \( T \) has a directed path \( P \) that visits all the nodes of \( T \).

Hint: Prove by contradiction. Consider a longest directed path in \( T \) and suppose that this path doesn’t visit all nodes in \( T \). What happens then?
Bonus Question

The degree $d(v)$ of a node $v \in V$ of an undirected graph $G = (V, E)$ is the number of its neighbors, i.e,

$$d(v) = | \{ u \in V \mid \{ v, u \} \in E \} |.$$

Show that the number of vertices of odd degree must be even.

*Hint: Recall the Handshaking lemma* \( \sum_{v \in V} d(v) = 2|E| \).