Exercise 1: Propositional Logic: Basic Terms

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation $I$ can be extended to composite formulae $\varphi$ over $\Sigma$ (cf. lecture). We write $I \models \varphi$ if $\varphi$ evaluates to $T$ (true) under $I$. In case $I \models \varphi$, $I$ is called a model for $\varphi$.

For each of the following formulae, give all interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 := (p \land \lnot q) \lor (\lnot p \lor q)$
(b) $\varphi_2 := (\lnot p \land (\lnot p \lor q)) \iff (p \land \lnot q)$
(c) $\varphi_3 := (p \land \lnot q) \rightarrow \lnot(p \land q)$
(d) $\varphi_4 := (p \land q) \rightarrow (p \lor r)$

Remark: $a \rightarrow b :\equiv \lnot a \lor b$, $a \leftrightarrow b :\equiv (a \rightarrow b) \land (b \rightarrow a)$, $a \not\rightarrow b :\equiv \lnot(a \rightarrow b)$.

Exercise 2: CNF and DNF

(a) Convert $\varphi_1 := (p \rightarrow q) \rightarrow (\lnot r \land q)$ into Conjunctive Normal Form (CNF).
(b) Convert $\varphi_2 := \lnot((\lnot p \rightarrow \lnot q) \land \lnot r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Exercise 3: Logical Entailment

A knowledge base $KB$ is a set of formulae over a given set of atoms $\Sigma$. An interpretation $I$ of $\Sigma$ is called a model of $KB$, if it is a model for all formulae in $KB$. A knowledge base $KB$ entails a formula $\varphi$ (we write $KB \models \varphi$), if all models of $KB$ are also models of $\varphi$.

Let $KB := \{p \lor q, \lnot r \lor p\}$. Show or disprove that $KB$ logically entails the following formulae.

(a) $\varphi_1 := (p \land q) \lor \lnot(\lnot r \lor p)$
(b) $\varphi_2 := (q \leftrightarrow r) \rightarrow p$
Exercise 4: Inference Rules and Calculi

Let \( \varphi_1, \ldots, \varphi_n, \psi \) be propositional formulae. An inference rule

\[
\frac{\varphi_1, \ldots, \varphi_n}{\psi}
\]

means that if \( \varphi_1, \ldots, \varphi_n \) are 'considered true', then \( \psi \) is 'considered true' as well (\( n = 0 \) is the special case of an axiom). A (propositional) calculus \( \mathcal{C} \) is described by a set of inference rules.

Given a formula \( \psi \) and knowledge base \( KB := \{ \varphi_1, \ldots, \varphi_n \} \) (where \( \varphi_1, \ldots, \varphi_n \) are formulae) we write \( KB \vdash_{\mathcal{C}} \psi \) if \( \psi \) can be derived from \( KB \) by starting from a subset of \( KB \) and repeatedly applying inference rules from the calculus \( \mathcal{C} \) to 'generate' new formulae until \( \psi \) is obtained.

Consider the following two calculi, defined by their inference rules (\( \varphi, \psi, \chi \) are arbitrary formulae).

\[
\begin{align*}
\mathcal{C}_1: & \quad \varphi \to \psi, \psi \to \chi \quad \varphi \to \psi, \psi \to \varphi \\
& \quad \varphi \to \chi, \neg \varphi \to \psi, \varphi \leftrightarrow \psi
\end{align*}
\]

\[
\begin{align*}
\mathcal{C}_2: & \quad \varphi, \varphi \to \psi, \varphi \wedge \psi, (\varphi \wedge \psi) \to \chi \\
& \quad \varphi, \psi, \varphi \to (\psi \to \chi)
\end{align*}
\]

Using the respective calculus, show the following derivations (document your steps).

(a) \( \{ p \leftrightarrow \neg r, \neg q \to r \} \vdash_{\mathcal{C}_1} p \to q \)

(b) \( \{ p \wedge q, p \to r, (q \wedge r) \to s \} \vdash_{\mathcal{C}_2} s \)

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.