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Theoretical Computer Science - Bridging Course
Winter Term 2020/21
Exercise Sheet 9

for getting feedback submit electronically by 12:15, Monday, January 18, 2021

Exercise 1: Propositional Logic: Basic Terms

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \land \neg q) \lor (\neg p \lor q)$

(b) $\varphi_2 = (\neg p \land (\neg p \lor q)) \leftrightarrow (p \lor \neg q)$

(c) $\varphi_3 = (p \land \neg q) \to \neg(p \land q)$

(d) $\varphi_4 = (p \land q) \to (p \lor r)$

Remark: $a \to b :\equiv \neg a \lor b$, $a \leftrightarrow b :\equiv (a \to b) \land (b \to a)$, $a \not\to b :\equiv \neg(a \to b)$.

Exercise 2: CNF and DNF

(a) Convert $\varphi_1 := (p \to q) \to (\neg r \land q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \to \neg q) \land \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Exercise 3: Logical Entailment

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \lor q, \neg r \lor p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \land q) \lor \neg(\neg r \lor p)$

(b) $\varphi_2 := (q \leftrightarrow r) \to p$

Exercise 4: Inference Rules and Calculi

Let $\varphi_1, \dots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \dots, \varphi_n}{\psi}$$

means that if $\varphi_1, \dots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \dots, \varphi_n\}$ (where $\varphi_1, \dots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\begin{array}{ll} \mathbf{C_1}: & \dfrac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \dfrac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \dfrac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi} \\ \mathbf{C_2}: & \dfrac{\varphi, \varphi \to \psi}{\psi}, \dfrac{\varphi \wedge \psi}{\varphi, \psi}, \dfrac{(\varphi \wedge \psi) \to \chi}{\varphi \to (\psi \to \chi)} \end{array}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \wedge q, p \to r, (q \wedge r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.