Exercise 1: Proof by Induction

Prove by induction for any positive integer number \(n\), \(n^3 + 2n\) is divisible by 3.

Solution:

We prove the claim by induction on \(n\). Obviously, the statement is true for the base case \(n = 1\) since \(1^3 + 2(1) = 3\) is divisible by 3. Then, we assume that the claim holds true for \(n = k\), where \(k > 0\), i.e. \(k^3 + 2k\) is divisible by 3, which is equivalent to \(k^3 + 2k = 3m\), where \(m\) is a positive integer. It follows that

\[
(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + (3k^2 + 3k + 3) = 3m + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)
\]

which is divisible by 3. The third equality in the equation above uses our induction hypothesis. Hence, the statement also holds for \(n = k + 1\), ending our induction proof.

Exercise 2: Which Statement is True?

Let \(A, B\) and \(C\) be sets in some universal set \(U\). Which of the following statements is always true. Justify.

1. If \(A \cap B = A \cap C\), then \(B = C\).
2. If \(A \cup B = A \cup C\), then \(B = C\).
3. \(\overline{A \cup B} = \overline{A} \cap \overline{B}\). Remark: \(\overline{A}\) is the compliment of \(A\)

Solution:

1. False. Take \(A = \{1, 2, 3\}, B = \{1, 4\}\) and \(C = \{1, 5\}\). We have \(A \cap B = A \cap C\) and \(B \neq C\).
2. False. Take \(A = \{1, 2\}, B = \{1, 3\}\) and \(C = \{2, 3\}\). We have \(A \cup B = A \cup C\) and \(B \neq C\).
3. True (De Morgan’s law). Indeed,

\[
x \in \overline{A \cup B} \iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \iff x \in \overline{A} \text{ and } x \in \overline{B} \iff x \in \overline{A} \cap \overline{B}
\]

hence, \(\overline{A \cup B} = \overline{A} \cap \overline{B}\). (You can also use Venn diagram to prove it).
Exercise 3: Visiting All Nodes (Part 1)  
(3 Points)

A simple graph is a graph without self loops, i.e. every edge of the graph is an edge between two distinct nodes. A complete graph is a simple undirected graph in which every pair of distinct nodes is connected by a unique edge e.g. a triangle on 3 nodes.

Prove that every complete graph $G$ has a path $P$ that visits all the nodes of $G$.

Solution:

We prove existence of path $P$ by construction as follows. First, pick any node $u$ in $G$. Since $G$ is a complete graph, there exists an edge from $u$ to each one of the other nodes, different from $u$. Second, choose one of these nodes say $v$, hence $\{u, v\}$ is an edge in $G$ and will be the first edge in the construction of $P$. Then, delete $u$ from $G$ and name the new graph $G_1$. Notice $G_1$ is still a complete graph. Pick any node $x$ in $G_1$ other than $v$, hence $\{v, x\}$ is an edge in $G_1$, thus in $G$, and will be the second edge in the construction of $P$. Repeat this process until only one node is left. Therefore, path $P$ is constructed and visits all nodes of $G$.

Alternative approaches: You can also prove this by induction on the number of nodes of $G$ or contradiction. However with contradiction, you may have to use the same trick employed in the upcoming exercise: considering a longest path $P$ in $G$.

Exercise 4: Visiting All Nodes (Part 2)  
(7 Points)

A directed path $P$ on $n$ vertices is a simple directed graph whose edge set is the following set of ordered pairs $\{(v_i, v_{i+1}) \mid 1 \leq i \leq n - 1 \text{ and } v_i \text{ is a node in } P\}$ i.e. a path in which all the arrows point in the same direction as its steps. We write $P = v_1v_2...v_n$ to denote the directed path $P$.

A tournament is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a directed edge with any one of the two possible orientations.

Prove that every tournament $T$ has a directed path $P$ that visits all the nodes of $T$.

Hint: Prove by contradiction. Consider a longest directed path in $T$ and suppose that this path doesn’t visit all nodes in $T$. What happens then?

Solution:

Let $T = (V, E)$ be a a tournament on $n$ nodes and $P = v_1...v_s$ be a longest directed path in $T$, where $v_i \in V$ for $1 \leq i \leq s$.

If $s = n$, then we are done.

Else $s \neq n$, then there exists a node $u$ in $T$ and not in $P$. Notice that since $T$ is a complete graph, there is an edge from $u$ to $v_i$, for all $1 \leq i \leq s$. However, we still don’t know whether $(u, v_1)$ or $(v_1, u)$ is in $E$. Now, if $(u, v_1) \in E$, then $P' = u v_1v_2...v_s$ is a directed path longer than $P$, a contradiction. So, it is necessarily that $(v_1, u) \in E$. Similarly, we show that $(u, v_s) \in E$ necessarily.

Then, there exists $1 \leq i_0 \leq s$ such that $(v_{i_0}, u)$ and $(u, v_{i_0+1})$ are both in $E$. Therefore, path $P' = v_1...v_{i_0}uv_{i_0+1}...v_s$ is a directed path longer than $P$, a contradiction. Hence, $s = n$ i.e. $P$ is a directed path visiting all the nodes of $T$. 

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**Bonus Question**

The degree $d(v)$ of a node $v \in V$ of an undirected graph $G = (V, E)$ is the number of its neighbors, i.e.,

$$d(v) = |\{u \in V \mid \{v, u\} \in E\}|.$$

Show that the number of vertices of odd degree must be even.

*Hint: Recall the Handshaking lemma $\sum_{v \in V} d(v) = 2|E|$.***

**Solution**

We prove by contradiction. Suppose that the number of nodes with odd degrees is not even i.e. odd. Hence, $2|E| = \sum_{v \in V \text{ s.t. } v \text{ with even degree}} d(v) + \sum_{v \in V \text{ s.t. } v \text{ with odd degree}} d(v)$.

Thus, even = even + odd = odd, a contradiction.