Exercise 1: Another Decidability Question

Let $S = \{ \langle M \rangle \mid M \text{ is a DFA that accepts } w^R \text{ whenever it accepts } w \}$. Show that $S$ is a decidable language.

Remark: You can use that it is not difficult to construct a TM, which tests whether an input is the well formed encoding of a DFA.

Sample Solution

Define the reverse operation denoted by $R$ on any language $L$ as follows $L^R = \{ w^R \mid w \in L \}$. Notice that $\langle M \rangle \in S \iff L(M) = L(M)^R$. We construct a decider for $S$ in the following manner:

On input $\langle M_1 \rangle$, where $M_1$ is a DFA:

- Construct a DFA $M_2$ for the language $L(M_1)^R$.
- Run the decider $F$ (for the $EQ_{DFA}$ problem known from the lecture) on input $\langle M_1, M_2 \rangle$. Accept iff $F$ accepts.

Exercise 2: Big-O Notation

The set $O(f)$ contains all functions that are asymptotically not growing faster than the function $f$ (when additive or multiplicative constants are neglected). That is:

$$g \in O(f) \iff \exists c \geq 0, \exists M \in \mathbb{N}, \forall n \geq M : g(n) \leq c \cdot f(n)$$

For the following pairs of functions, check whether $f \in O(g)$ or $g \in O(f)$ or both. Prove your claims (you do not have to prove a negative result $\notin$, though).

(a) $f(n) = 20n$, $g(n) = 0.1 \cdot n^2$
(b) $f(n) = \sqrt[3]{n^2}$, $g(n) = \sqrt{n}$
(c) $f(n) = \log_2(2^n \cdot n^3)$, $g(n) = 3n$  

Hint: You may use that $\log_2 n \leq n$ for all $n \in \mathbb{N}$. 

Sample Solution

(a) It is $20n \in O(0.1n^2)$. To show that we require constants $c, M$ such that $20n \leq c \cdot 0.1n^2$ for all $n \geq M$. Obviously this is the case for $c = 200$ and $M = 1$.

(b) We have $g(n) \in O(f(n))$. Let $c := 1$ and $M := 1$. Then we have

\[ g(n) \leq c \cdot f(n) \]  \hspace{1cm} (1)
\[ \Leftrightarrow \quad \sqrt{n} \leq n^{2/3} \]  \hspace{1cm} (2)
\[ \Leftrightarrow \quad 1 \leq n^{1/6} \]  \hspace{1cm} (3)
\[ \Leftrightarrow \quad 1 \leq n \]  \hspace{1cm} (4)

The last inequality is satisfied because $n \geq M = 1$.

(c) $f(n) \in O(g(n))$ holds. We give $c > 0$ and $M \in \mathbb{N}$ such that for all $n \geq M : \log_2(2^n \cdot n^3) \leq c \cdot n$. As $cn \in O(g(n))$ holds for every constant $c > 0$ the result will follow with the transitivity of the $O$-notation.

\[ \log_2(2^n \cdot n^3) \]
\[ = \log_2(2^n) + \log_2(n^3) \]
\[ = n + 3 \cdot \log_2(n) \]
\[ \leq n + 3n = 4n. \]

Thus $\log_2(2^n \cdot n^3) \leq c \cdot n$ for $n \geq M := 1$ and $c := 4$.

We also have that $g(n) \in O(f(n))$ holds because

\[ g(n) = 3n \leq 3(n + 3 \cdot \log_2(n)) = 3(\log_2(2^n \cdot n^3)) = 3 \cdot f(n). \]

Thus with $c = 3$ and for $n \geq M := 1$ we have $g(n) \leq cf(n)$.

Exercise 3: Sorting Functions by Asymptotic Growth

Sort the following functions by asymptotic growth using the $O$-notation. Write $g <_O f$ if $g \in O(f)$ and $f \notin O(g)$. Write $g =_O f$ if $f \in O(g)$ and $g \in O(f)$.

<table>
<thead>
<tr>
<th>Function</th>
<th>$n^2$</th>
<th>$\sqrt{n}$</th>
<th>$n^{100}$</th>
<th>$2^n$</th>
<th>$\log(n^2)$</th>
<th>$(\log n)^2$</th>
<th>$\log(\sqrt{n})$</th>
<th>$n!$</th>
<th>$\log n$</th>
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</thead>
<tbody>
<tr>
<td>$3n$</td>
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<td>$\log n$</td>
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<td>$n \cdot 2^n$</td>
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Sample Solution

<table>
<thead>
<tr>
<th>Function</th>
<th>$\sqrt{\log n}$</th>
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<th>$\log(\sqrt{n})$</th>
<th>$n$</th>
<th>$n^{100}$</th>
<th>$n^2$</th>
<th>$n^{100}$</th>
<th>$n!$</th>
<th>$n^3$</th>
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</thead>
<tbody>
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<td>$n^3$</td>
<td>$n^n$</td>
</tr>
</tbody>
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