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Algorithms and Datastructures
Winter Term 2021/2022
Sample Solution Exercise Sheet 3

Exercise 1: Bucket Sort

Bucketsort is an algorithm to stably sort an array $A[0..n{-}1]$ of n elements where the sorting keys of the elements take values in $\{0, \ldots, k\}$. That is, we have a function `key` assigning a key $\texttt{key}(x) \in \{0, \ldots, k\}$ to each $x \in A$.

The algorithm works as follows. First we construct an array $B[0..k]$ consisting of (initially empty) FIFO queues. That is, for each $i \in \{0, \ldots, k\}$, $B[i]$ is a FIFO queue. Then we iterate through A and for each $j \in \{0, \ldots, n-1\}$ we attach $A[j]$ to the queue $B[\texttt{key}(A[j])]$ using the function `enqueue`.

Finally we empty all queues $B[0], ..., B[k]$ using `dequeue` and write the returned values back to A, one after the other. After that, A is sorted with respect to `key` and elements $x, y \in A$ with $\texttt{key}(x) = \texttt{key}(y)$ are in the same order as before.

Implement *Bucketsort* based on this description. You can use the template `BucketSort.py` which uses an implementation of FIFO queues that are available in `Queue.py` and `ListElement.py`.[1] An example of usage of this template is the following:

```
from Queue import Queue
from ListElement import ListElement
q = Queue()
q.enqueue(ListElement(5))
q.enqueue(ListElement(17))
q.enqueue(ListElement(8))
while not q.is_empty():
    print(q.dequeue().get_key())
```

This would print the numbers $5, 17, 8$ on three separate lines.

Solution:

```
def bucket_sort(array, k, key=lambda x: x):
    '''
    Implements the bucket sort algorithm to sort
    data elements using a key function to
    assign sorting keys to data elements

    Args:
    array: array of data elements
    k: largest key
    key: a function mapping data elements to values
    in range(k+1) (idendity function as default)
```

[1] Remember to make unit-tests and to add comments to your source code.

```
>>> bucket_sort([210,121,203,420,307],2,lambda x: int(x / 10) % 10)
[203, 307, 210, 121, 420]
>>> bucket_sort([], 10)
[]
>>> bucket_sort([10-i for i in range(10)], 10)
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
'''
# add your code here
bucket = [Queue() for i in range(k+1)]
for i in range(len(array)):
    bucket[key(array[i])].enqueue(ListElement(array[i]))
i = 0
for j in range(k+1):
    while not bucket[j].is_empty():
        array[i] = bucket[j].dequeue().get_key()
        i += 1
return array
```

Exercise 2: Radix Sort

Assume we want to sort an array $A[0..n-1]$ of size n containing integer values from $\{0, \dots, k\}$ for some $k \in \mathbb{N}$. We describe the algorithm *Radixsort* which uses `BucketSort` as a subroutine.
Let $m = \lfloor \log_b k \rfloor$. We assume each key $x \in A$ is given in base-b representation, i.e., $x = \sum_{i=0}^{m} c_i \cdot b^i$ for some $c_i \in \{0, \dots, b-1\}$. First we sort the keys according to c_0 using `BucketSort`, afterwards we sort according to c_1 and so on.[2]

(a) Implement *Radixsort* based on this description. You may assume $b = 10$, i.e., your algorithm should work for arrays containing numbers in base-10 representation. Use *Bucketsort* as a subroutine.

(b) Compare the runtimes of *Bucketsort* and *Radixsort*. For both algorithms and each $k \in \{i \cdot 10^4 \mid i = 1, \dots, 50\}$, use an array of size 10^4 with randomly chosen keys from $\{0, \dots, k\}$ as input and plot the runtimes. Shortly discuss your results.

(c) Explain the asymptotic runtime of your implementations of *Bucketsort* and *Radixsort* depending on n and k.

Solution:

(a)
```
def radix_sort(array, k):
    '''
    Implements the radix sort algorithm to sort
        data elements with keys in range(k+1)
    Args:
        array: array of data elements
        k: largest key
    >>> radix_sort([123,1111,789,456,0,12,13,247],2000)
    [0, 12, 13, 123, 247, 456, 789, 1111]
    >>> radix_sort([1000-i for i in range(0,1000)],1000) == \
        [i for i in range(1,1001)]
    True
    '''
```

[2] The i-th digit c_i of a number $x \in \mathbb{N}$ in base-b representation (i.e, $x = c_0 \cdot b^0 + c_1 \cdot b^1 + c_2 \cdot b^2 + \dots$), can be obtained via the formula $c_i = (x \texttt{ mod } b^{i+1}) \texttt{ div } b^i$, where `mod` is the modulo operation and `div` the integer division.

```python
    m = math.ceil(math.log(k, 10))
    for i in range(m+1):
        key = lambda x: (x % 10**(i+1)) // 10**i
        BucketSort.bucket_sort(array, 10, key)
    return array
```

(b) See Figure 1. We see that *Bucketsort* is linear in k. For *Radixsort* the situation is not that clear. At the first sight, the runtime could be constant, but upon closer examination (see Figure 2) we see a step at $k = 10^5$. The reason is that *Radixsort* calls *Bucketsort* for each digit in the input and the number of these digits (and therefore the calls of *Bucketsort*) is increased from 5 to 6 at $k = 10^5$.

(c) *Bucketsort* goes through A twice, once to write all values from A into the buckets and another time to write the values back to A. This takes time $\mathcal{O}(n)$ as writing a value into a bucket and from a bucket back to A costs $\mathcal{O}(1)$. Additionally, *Bucketsort* needs to allocate k empty lists and write it into an array of size k which takes time $\mathcal{O}(k)$. Hence, the runtime is $\mathcal{O}(n + k)$.

RadixSort calls *Bucketsort* for each digit. The keys have $m = \mathcal{O}(\log k)$ digits, so we call *Bucketsort* $\mathcal{O}(\log k)$ times. One run of *Bucketsort* takes $\mathcal{O}(n)$ here as the keys according to which *Bucketsort* sorts the elements are from the range $\{0, \ldots, 9\}$. The overall runtime is therefore $\mathcal{O}(n \log k)$.

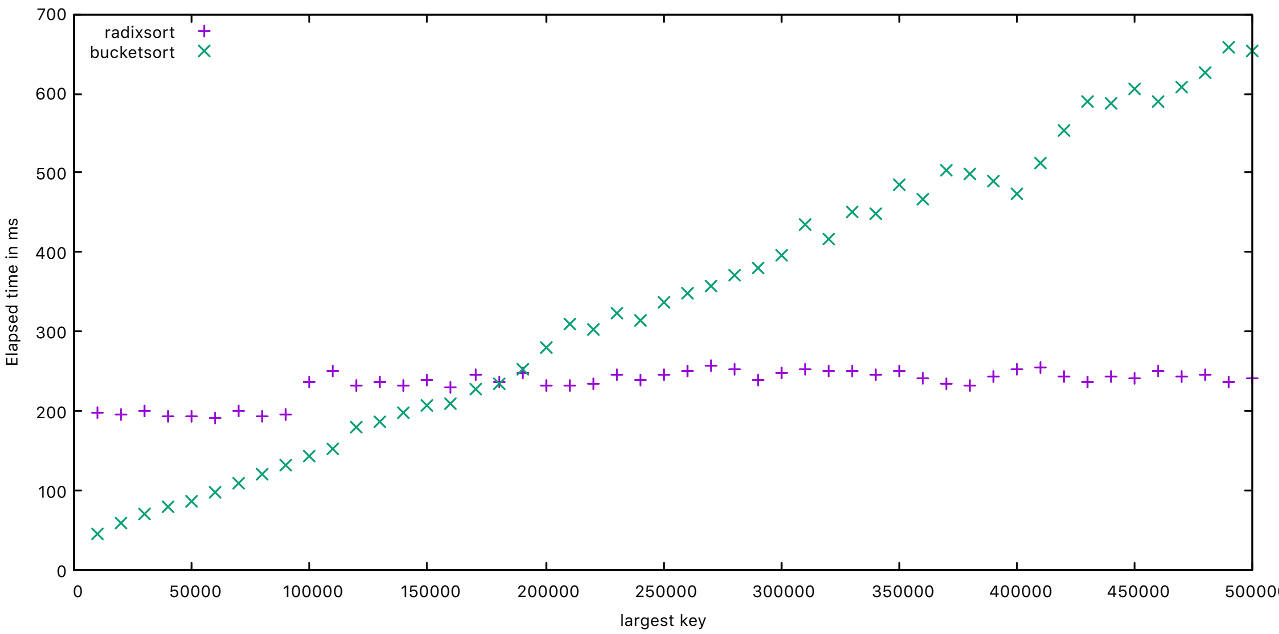

Figure 1: Plot for exercise 2 b).

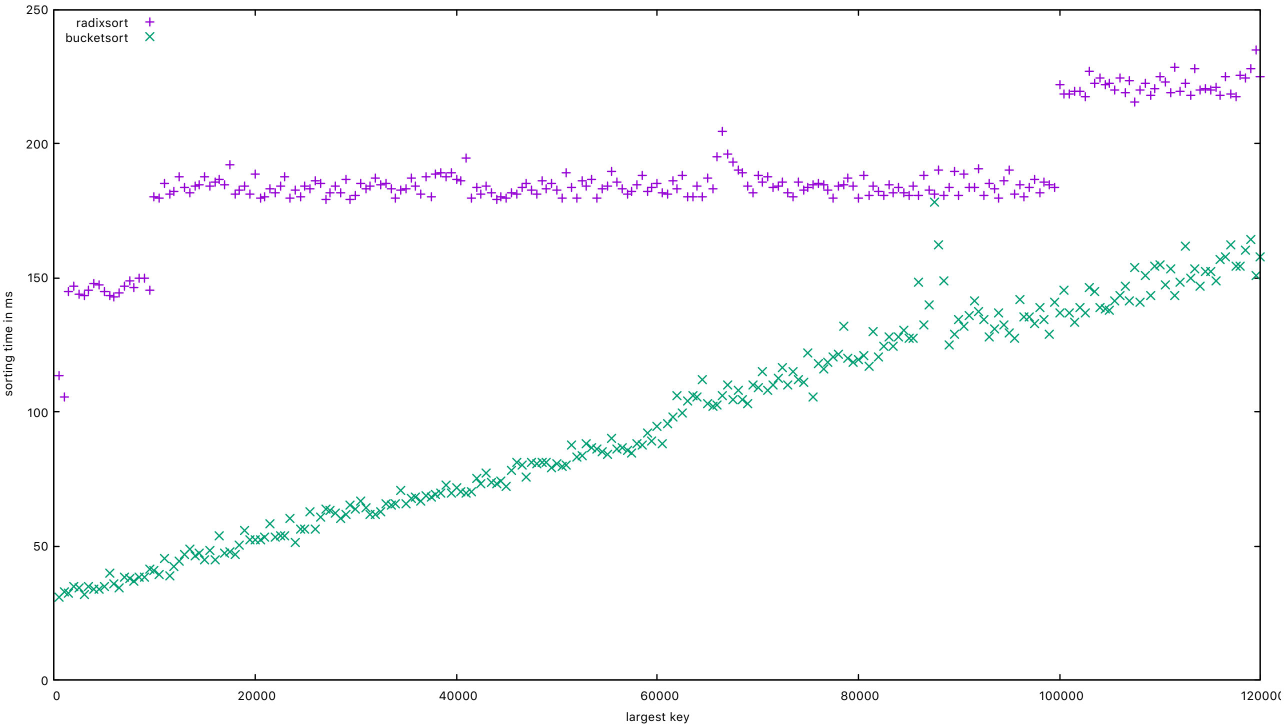

Figure 2: Considering a larger range of keys to visualize the second step at 10^6.