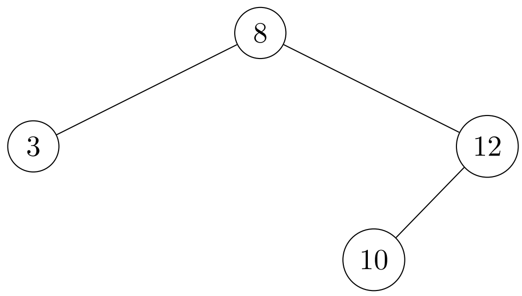
University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn

Algorithms and Data Structures
Winter Term 2021/2022
Sample Solution Exercise Sheet 6

Exercise 1: Binary Search Trees I

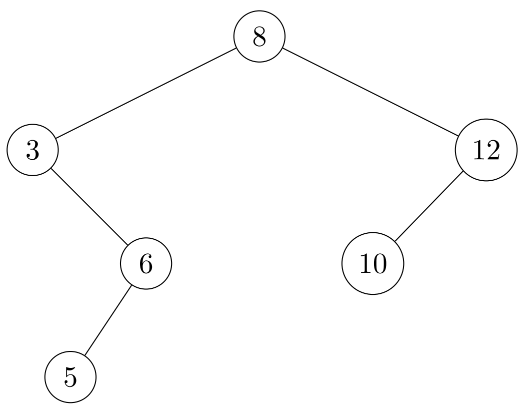
Consider the following binary search tree.



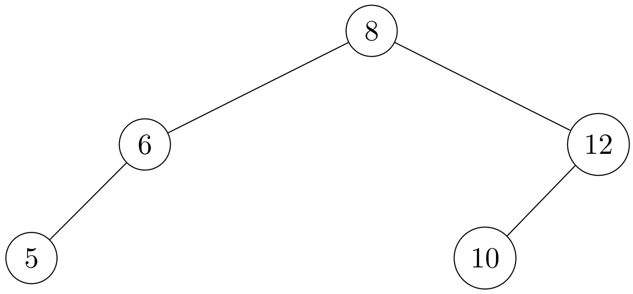
1. Give all sequences of `insert`(key) operations that generate the tree.
2. Draw the tree after the following sequence of operations: `insert`(6), `insert`(5), `remove`(3).

Sample Solution

1. (i) `insert`(8),`insert`(3), `insert`(12), `insert`(10)
   (ii) `insert`(8),`insert`(12), `insert`(3), `insert`(10)
   (iii) `insert`(8),`insert`(12), `insert`(10), `insert`(3)
2. After `insert`(6) and `insert`(5):



After `remove`(3):



Exercise 2: Binary Search Trees II

(a) Describe a function that takes a binary search tree B and a key x as input and generates the following output:

- If there is an element v in B with $v.key = x$, return v.
- Otherwise, return the pair (u, w) where u is the tree element with the next smaller key and w is the element with the next larger key. It should be $u =$ None if x is smaller than any key in the tree and $w =$ None if x is larger than any key in the tree.

For your description you can use pseudo code or a sufficiently detailed description in English.

Analyze the runtime of your function.

(b) Describe a function which returns the depth of a binary search tree and analyze the runtime.

(c) Describe a function that for a given binary search tree with n nodes and a given $k \leq n$ returns a list with the k smallest keys from the tree. Analyze the runtime.

Sample Solution

(a) **Algorithm 1** `return-closest`(x)

$v \leftarrow$ `find`(x)
if $v \neq$ None **then**
    return v
else
    `insert`(x)
    $(p, s) \leftarrow$ (`pred`(x), `succ`(x))
    `delete`(x)
    return (p, s)

All subprocedures that we call (`find`, `insert`, `pred`, `succ`) are known from the lecture and take $\mathcal{O}(d)$ with d being the depth of the tree. So the overall runtime is $\mathcal{O}(d)$.

(b) We can do a recursive traversal of the tree where we keep track of the current recursion depth. Then a call of `depth`(r) on the root r ot the BST returns its depth.

Algorithm 2 `depth`(v)

if $v =$ None **then**
    return -1   ▷ *depth of a childless node must be 0, hence we define the depth of None as -1*
else **return** max(`depth`(v.`left`)+1, `depth`(v.`right`)+1)

The runtime corresponds to the runtime of the traversal of the whole tree which is $\mathcal{O}(n)$ as we have just one recursive call for each node and each recursive call costs $\mathcal{O}(1)$ (c.f., pre-, in-, post-order traversal algorithms given in the lecture).

As an alternative solution, we can run a BFS which takes $\mathcal{O}(n)$. If v is the node visited last by the BFS, do

Algorithm 3 `traverse-up`(v)

$d \leftarrow 0$
while v.`parent` $\neq$ None **do**
  $d \leftarrow d + 1$
  $v \leftarrow v$.`parent`
return d

This takes $\mathcal{O}(d)$ where d is the depth of the tree. Since $d \leq n$ the overall runtime is $\mathcal{O}(n+d) = \mathcal{O}(n)$.

(c) Initialize an empty list K. We roughly do the following. Make an in-order traversal of the tree and each time visiting a node, add it to K. Stop if $|K| \geq k$. The following pseudocode formalizes this.

Algorithm 4 `inorder_variant`(node) ▷ Assume list K is given globally, initially empty

if node $\neq$ `None` **then**
  `inorder_variant`(node.`left`)
  if $|K| \geq k$ **then**
    return
  K.append($node.key$)
  `inorder_variant`(node.`right`)

The runtime is $\mathcal{O}(d + k)$ where d is the depth of the tree. We prove this in the following.

Let K be the set of k nodes representing the k smallest keys in the BST. Obviously, the in-order traversal must visit all nodes in K once. In accorddance with the lecture a call of `inorder_variant`(root) adds all keys in ascending order to K.

Let A be the set of nodes in the BST which are not in K but in which a recursive call will be made. Since the recursion is aborted (with the `return` statement) after reporting k nodes, the set A contains exactly the nodes which are ancestors of a node in K, but are not in K themselves. Since the runtime of a single recursive call (neglecting subcalls) is $\big(1\big)$ the total runtime is $\mathcal{O}(|A| + |K|)$.

By definition we have $|K| = k$, so it remains to determine the size of A. We claim that all nodes in A are on a path from the root to a leaf, that is, $|A| \leq d$. This is the case if there do not exist two nodes in A so that neither is an ancestor of the other.

For a contradiction, suppose that two such nodes u, v exist so that neither u is ancestor of v nor vice versa. Assume (without loss of generality) that `key`$(u) \leq$ `key`(v). That means u is in the left and v is in the right subtree of some common ancestor a of u and v.

By definition v has a node $w \in K$ in its subtree. Since v is in the right subtree and u is in the left subtree of a, we have `key`$(w) \geq$ `key`(u) and w has a higher in-order-position. But then we would have $u \in K$ as well, a contradiction to $u \in A$.