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Theoretical Computer Science - Bridging Course
Exercise Sheet 1

Due: Sunday, 24th of October 2021, 23:59 pm

Exercise 1: Mathematical Proofs *(3+1+3+2 Points)*

1. Use mathematical induction to prove that for any positive integer n, $6^n - 1$ is divisible by 5.
   Remark: Integer k is divisible by integer ℓ if there exists an integer s such that $k = s \cdot \ell$.
2. Prove or disprove the following statement: any number written as $6n - 1$ is a prime number, where n is some positive integer.
3. Show that any prime number greater than 3 can be written of the form $6n + 1$ or $6n - 1$, where n is some positive integer.

   Hint: try to first check all possible forms that any integer number greater than 3 can be written as by using the following known result on some special case: given any two positive integers a and b, with $b \neq 0$, there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < b$.
4. Let A and B be two subsets of some universal set E. Prove the following :
   $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A} = E \backslash A$ is the compliment of A.

Exercise 2: Where is the Even Degree Node? *(4 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ in an undirected graph $G = (V, E)$ is the number of its neighbors, i.e,

$$d(v) = |\{u \in V \mid \{v, u\} \in E\}|.$$

Show that every simple graph with an odd number of nodes contains a node with even degree.

Hint: Consider the sum $D = \sum_{v \in V} d(v)$ of all degrees.

Exercise 3: More on Graphs *(3+4 Points)*

A graph $G = (V, E)$ is said to be connected if for every pair of vertices $u, v \in V$ such that $u \neq v$ there exists a path in G connecting u to v.

1. Prove that if G is connected, then for any two non empty subsets V_1 and V_2 of V such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, there exists an edge joining a vertex in V_1 to a vertex in V_2.
2. Let G be a simple, connected graph and P be a path of the longest length ℓ in G. Show that if the two ends of P are adjacent, then $V = V(P)$, where $V(P)$ is the set of vertices of P.

Bonus Question: A tree is a simple, connected graph without cycles. Prove that every tree with $n \geq 1$ nodes has $n - 1$ edges. *Hint: You can use the fact that every tree T contains a vertex v of degree* 1.