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Algorithm Theory
Sample Solution Bonus Sheet

Due: Friday, 17th of January 2024, 10:00 am

Bonus Points: The points achievable in this exercise sheet will not increase the threshold of points needed.

Exercise 1: Edge-Coloring *(8 Bonus Points)*

We define a *proper edge-coloring* of a graph as an assignment of 'colors' to the edges of the graph so that no two incident edges have the same color. For example, if the graph is a path, there is a proper edge-coloring with two colors as one can color the edges in alternating fashion from left to right.
For some $d \in \mathbb{N}$, we call a simple undirected graph d-regular if each node has exactly d neighbors. Let $G_d = (A \dot\cup B, E)$ be a d-regular bipartite graph.

(a) Show that there exists a perfect matching in G_d. *(4 Points)*

(b) Show that there exists a *proper edge-coloring* with colors $\in \{1, \dots, d\}$ in G_d. *(4 Points)*

Sample Solution

(a) We know that $|E| = |A| \cdot d$ as well as $|E| = |B| \cdot d$. Combining these terms we have $|A| = |E|/d = (|B| \cdot d)/d = |B|$. By Hall's Theorem, the statement of the task holds if $|A| = |B|$ and $\forall A' \subseteq A : |N(A')| \geq |A'|$. The first part is already shown. Let $E_{A'}$ be the edges of G_d starting from a node in A' (and thus $|E_{A'}| = |A'| \cdot d$). Since each node in $N(A') \subseteq B$ has at most d edges coming from a node in A', we have $|N(A')| \geq |E_{A'}|/d = |A'|$.

(b) We show the existence of the edge-coloring by induction over d. The base case is $d = 1$. In G_1 all edges are perfect matching edges, and hence not incident to each other. Thus, labeling all edges with 'color' 1 fulfills the proper edge-coloring requirements. Now, for the induction step let's show that G_{d+1} can be edge-colored with $d+1$ colors if every graph G_d is d edge colorable (hypothesis). Let us first compute a perfect matching on G_{d+1}. We label all edges in the matching with color $d+1$ (since they are in a perfect matching no edges with label $d+1$ are incident). Deleting these edges, the remaining graph is a bipartite d-regular graph, that can be properly edge-colored with colors $1, \dots, d$ by our hypothesis. Hence, the edges of G_{d+1} are a composition of such a proper edge-coloring with d colors and the additional color $d+1$ of non-incident edges that gives us proper edge-coloring on G_{d+1}.

Exercise 2: Chess tournament *(6 Bonus Points)*

Assume that there are n chess players $1, \dots, n$ that you need to pair up for playing against each other in a chess tournament.
There are some players who must play their next game with the white pieces and there are some players who must play their next game with the black pieces. There are also players for whom it does not matter if they play with the white or the black pieces.
In addition, each player i has a rating value r_i, which is a positive integer.

Each chess game in the tournament must be played between exactly two players: one playing with the white pieces and the other with the black pieces. Further, each player should play in at most one game. Additionally, to ensure balanced games, the absolute difference in rating between the two players in a game must be smaller than 100.

a) Describe a polynomial-time algorithm to determine a largest possible set of chess games that can be arranged with the available n players. You can use algorithms from the lecture as a black box. *(3 Points)*

b) Assume that we make the (strange) requirement that the absolute difference in rating between the two players of a game must be an odd number < 100? Argue why the problem of determining a maximum set of possible chess games now becomes easier! *(3 Points)*

Sample Solution

(a) We construct a graph $G = (V, E)$, where we add a node for every player. We add an edge between two nodes if the rating is smaller than 100 and if either one of them is a 'white' and the other a 'black' player or if at least one of them can play with both colors. A matching on G solves the problem. To compute it, we can use the Edmond's Blossom Algorithm as this runs in time $O(mn^2)$.

(b) The graph G, as constructed above plus the new requirement, always constructs a bipartite graph. On the 'left' side are the players with even ratings and on the 'right' there are players with odd ratings. There can not exist and edge on the left side as the difference of their ratings would be even and for the same reason there can not exist and edge between nodes on the right side. This problem can be solved in time $O(m \cdot n)$.