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Algorithm Theory
Sample Solution Exercise Sheet 2

Due: Monday, 4th of November 2024, 10:00 am

Assumption: You may assume that calculations with real numbers can be performed with arbitrary precision in constant time.

Exercise 1: Faster Polynomial Multiplication (5 Points)

Let $p(x) := 4x^3 - 2x + 1$ and $q(x) := -3x^2 + x + 5$. The goal is to compute $p(x) \cdot q(x)$ with the help of the FFT algorithm. Please, make use of the following sketch:

1. Illustrate the **divide** procedure of the algorithm (for both functions p and q). More precisely, for the i-th divide step (with focus on $p(x)$), write down all the polynomials p_{ij} for $j \in \{0, ..., 2^i - 1\}$ that you obtain from further dividing the polynomials from the previous divide step $i - 1$ (we define $p_{00} := p$, and the first split is into p_{10} and p_{11} and so on...).
2. Illustrate the **combine** procedure of the algorithm (for both functions p and q). That is, starting with the polynomials of the smallest degree as base cases, compute the DFT of p_{ij} (respectively q_{ij}) bottom up with the recursive formula given in the lecture.
3. **Multiply** the polynomials. More specific, give the point value representation of $p(x) \cdot q(x)$.
4. Use the **inverse** DFT procedure from the lecture to get the final coefficients for $p(x) \cdot q(x)$.

Write down all intermediate results to get partial points in the case of a typo.

Sample Solution

1. Note that for the divide step we want to preserve that $p(x) = p_0(x^2) + x \cdot p_1(x^2)$ where p_0 contains the even coefficients and p_1 the odds.
   divide p**:**

$$\begin{aligned} p(x) &= 4x^3 - 2x + 1 \\ p_0(x) &= 1 \\ p_1(x) &= -2 + 4x \\ p_{10}(x) &= -2 \\ p_{11}(x) &= 4 \end{aligned}$$

   divide q**:**

$$\begin{aligned} q(x) &= -3x^2 + x + 5 \\ q_0(x) &= 5 - 3x \\ q_1(x) &= 1 \\ q_{00}(x) &= 5 \\ q_{01}(x) &= -3 \end{aligned}$$

2. In the combine step we compute the required values in a bottom-up fashion using the following formula from the lecture:

$$p(w_N^k) := \begin{cases} p_0\left(w_{N/2}^k\right) + w_N^k \cdot p_1\left(w_{N/2}^k\right) & \text{if } k < N/2 \\ p_0\left(w_{N/2}^{k-N/2}\right) + w_N^k \cdot p_1\left(w_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

combine p**:**

$$\begin{aligned} p_1(w_4^0) &= p_{10}(w_2^0) + w_4^0 \cdot p_{11}(w_2^0) = -2 + 1 \cdot (4) = 2 \\ p_1(w_4^1) &= p_{10}(w_2^1) + w_4^1 \cdot p_{11}(w_2^1) = -2 + i \cdot (4) = -2 + 4i \\ p_1(w_4^2) &= p_{10}(w_2^0) + w_4^2 \cdot p_{11}(w_2^0) = -2 - 1 \cdot (4) = -6 \\ p_1(w_4^3) &= p_{10}(w_2^1) + w_4^3 \cdot p_{11}(w_2^1) = -2 - i \cdot (4) = -2 - 4i \end{aligned}$$

$$\begin{aligned} p_0(w_4^0) &= 1 \\ p_0(w_4^1) &= 1 \\ p_0(w_4^2) &= 1 \\ p_0(w_4^3) &= 1 \end{aligned}$$

Now we can go to the next recursion level. Note that we have $w_8^0 = 1$, $w_8^1 = \frac{1+i}{\sqrt{2}}$, $w_8^2 = i$, $w_8^3 = \frac{-1+i}{\sqrt{2}}$, $w_8^4 = -1$, $w_8^5 = -w_8^1$, $w_8^6 = -i$, $w_8^7 = -w_8^3$.

$$\begin{aligned} p(w_8^0) &= p_0(w_4^0) + w_8^0 \cdot p_1(w_4^0) = 1 + 2 = 3 \\ p(w_8^1) &= p_0(w_4^1) + w_8^1 \cdot p_1(w_4^1) = 1 - (3 - i) \cdot \sqrt{2} \\ p(w_8^2) &= p_0(w_4^2) + w_8^2 \cdot p_1(w_4^2) = 1 - 6i \\ p(w_8^3) &= p_0(w_4^3) + w_8^3 \cdot p_1(w_4^3) = 1 - (3 + i) \cdot \sqrt{2} \\ p(w_8^4) &= p_0(w_4^0) - w_8^0 \cdot p_1(w_4^0) = -1 \\ p(w_8^5) &= p_0(w_4^1) - w_8^1 \cdot p_1(w_4^1) = 1 + (3 - i) \cdot \sqrt{2} \\ p(w_8^6) &= p_0(w_4^2) - w_8^2 \cdot p_1(w_4^2) = 1 + 6i \\ p(w_8^7) &= p_0(w_4^3) - w_8^3 \cdot p_1(w_4^3) = 1 - (3 + i) \cdot \sqrt{2} \end{aligned}$$

combine q**:**

$$\begin{aligned} q_0(w_4^0) &= q_{00}(w_2^0) + w_4^0 \cdot q_{01}(w_2^0) = 5 + 1 \cdot -3 = 2 \\ q_0(w_4^1) &= q_{00}(w_2^1) + w_4^1 \cdot q_{01}(w_2^1) = 5 + i \cdot -3 = 5 - 3i \\ q_0(w_4^2) &= q_{00}(w_2^0) + w_4^2 \cdot q_{01}(w_2^0) = 5 - 1 \cdot -3 = 8 \\ q_0(w_4^3) &= q_{00}(w_2^1) + w_4^3 \cdot q_{01}(w_2^1) = 5 - i \cdot -3 = 5 + 3i \end{aligned}$$

$$\begin{aligned} q_1(w_4^0) &= 1 \\ q_1(w_4^1) &= 1 \\ q_1(w_4^2) &= 1 \\ q_1(w_4^3) &= 1 \end{aligned}$$

$$
\begin{aligned}
q(w_8^0) &= q_0(w_4^0) + w_8^0 \cdot q_1(w_4^0) = 3 \\
q(w_8^1) &= q_0(w_4^1) + w_8^1 \cdot q_1(w_4^1) = (5 - 3i) + (1 + i)/\sqrt{2} \\
q(w_8^2) &= q_0(w_4^2) + w_8^2 \cdot q_1(w_4^2) = 8 + i \\
q(w_8^3) &= q_0(w_4^3) + w_8^3 \cdot q_1(w_4^3) = (5 + 3i) + (-1 + i)/\sqrt{2} \\
q(w_8^4) &= q_0(w_4^0) - w_8^0 \cdot q_1(w_4^0) = 1 \\
q(w_8^5) &= q_0(w_4^1) - w_8^1 \cdot q_1(w_4^1) = (5 - 3i) - (1 + i)/\sqrt{2} \\
q(w_8^6) &= q_0(w_4^2) - w_8^2 \cdot q_1(w_4^2) = 8 - i \\
q(w_8^7) &= q_0(w_4^3) - w_8^3 \cdot q_1(w_4^3) = (5 - 3i) - (-1 + i)/\sqrt{2}
\end{aligned}
$$

3. **Multiply:**

$$
\begin{aligned}
p_{00}(w_8^0) \cdot q_{00}(w_8^0) &= 9 \\
p_{00}(w_8^1) \cdot q_{00}(w_8^1) &= (1 - 5i) - (23 - 29i)/\sqrt{2} \\
p_{00}(w_8^2) \cdot q_{00}(w_8^2) &= 14 - 47i \\
p_{00}(w_8^3) \cdot q_{00}(w_8^3) &= (1 + 5i) + (23 + 29i)/\sqrt{2} \\
p_{00}(w_8^4) \cdot q_{00}(w_8^4) &= -1 \\
p_{00}(w_8^5) \cdot q_{00}(w_8^5) &= (1 - 5i) + (23 - 29i)/\sqrt{2} \\
p_{00}(w_8^6) \cdot q_{00}(w_8^6) &= 14 + 47i \\
p_{00}(w_8^7) \cdot q_{00}(w_8^7) &= (1 + 5i) - (23 + 29i)/\sqrt{2}
\end{aligned}
$$

4. **Inverse DFT:** To efficiently compute the inverse DFT, we again have to do some bottom-up computation, now based on the polynomial $f(x) := y_7x^7 + y_6x^6 + ... + y_0$, where the y_i values are the y-values in the point value representation of $p(x) \cdot q(x)$.

$$
\begin{aligned}
f_0 &= y_6x^3 + y_4x^2 + y_2x + y_0 \\
f_1 &= y_7x^3 + y_5x^2 + y_3x + y_1 \\
f_{00} &= y_4x + y_0 \\
f_{01} &= y_6x + y_2 \\
f_{10} &= y_5x + y_1 \\
f_{11} &= y_7x + y_3 \\
f_{000} &= y_0 = 9 \\
f_{001} &= y_4 = -1 \\
f_{010} &= y_2 = 14 - 47i \\
f_{011} &= y_6 = 14 + 47i \\
f_{100} &= y_1 = (1 - 5i) - (23 - 29i)/\sqrt{2} \\
f_{101} &= y_5 = (1 - 5i) + (23 - 29i)/\sqrt{2} \\
f_{110} &= y_3 = (1 + 5i) + (23 + 29i)/\sqrt{2} \\
f_{111} &= y_7 = (1 + 5i) - (23 + 29i)/\sqrt{2}
\end{aligned}
$$

$$
\begin{aligned}
f_{00}(w_2^0) &= f_{000}(w_1^0) + w_2^0 \cdot f_{001}(w_1^0) = 9 - 1 = 8 \\
f_{00}(w_2^1) &= f_{000}(w_1^0) + w_2^1 \cdot f_{001}(w_1^0) = 9 + 1 = 10 \\
f_{01}(w_2^0) &= f_{010}(w_1^0) + w_2^0 \cdot f_{011}(w_1^0) = (14 - 47i) + 1 * (14 + 47i) = 28 \\
f_{01}(w_2^1) &= f_{010}(w_1^0) + w_2^1 \cdot f_{011}(w_1^0) = (14 - 47i) + -1 * ((14 + 47i)) = -94i \\
f_{10}(w_2^0) &= f_{100}(w_1^0) + w_2^0 \cdot f_{101}(w_1^0) = (1 - 5i) - (23 - 29i)/\sqrt{2} + ((1 - 5i) + (23 - 29i)/\sqrt{2}) \\
&= 2 - 10i \\
f_{10}(w_2^1) &= f_{100}(w_1^0) + w_2^1 \cdot f_{101}(w_1^0) = (1 - 5i) - (23 - 29i)/\sqrt{2} + ((1 - 5i) - (23 - 29i)/\sqrt{2}) \\
&= -(23 - 29i)\sqrt{2} \\
f_{11}(w_2^0) &= f_{110}(w_1^0) + w_2^0 \cdot f_{111}(w_1^0) = (1 + 5i) + (23 + 29i)/\sqrt{2} + ((1 + 5i) - (23 + 29i)/\sqrt{2}) \\
&= 2 + 10i \\
f_{11}(w_2^1) &= f_{110}(w_1^0) + w_2^1 \cdot f_{111}(w_1^0) = (1 + 5i) + (23 + 29i)/\sqrt{2} - ((1 + 5i) - (23 + 29i)/\sqrt{2}) \\
&= (23 + 29i)\sqrt{2}
\end{aligned}
$$

$$
\begin{aligned}
f_0(w_4^0) &= f_{00}(w_2^0) + w_4^0 \cdot f_{01}(w_2^0) = 8 + 1(28) = 36 \\
f_0(w_4^1) &= f_{00}(w_2^1) + w_4^1 \cdot f_{01}(w_2^1) = 10 + i(-94i) = 104 \\
f_0(w_4^2) &= f_{00}(w_2^0) + w_4^2 \cdot f_{01}(w_2^0) = 8 - 1(28) = -20 \\
f_0(w_4^3) &= f_{00}(w_2^1) + w_4^3 \cdot f_{01}(w_2^1) = 10 - i(-94i) = -84
\end{aligned}
$$

$$
\begin{aligned}
f_1(w_4^0) &= f_{10}(w_2^0) + w_4^0 \cdot f_{11}(w_2^0) = 2 - 10i + 1(2 + 10i) = 4 \\
f_1(w_4^1) &= f_{10}(w_2^1) + w_4^1 \cdot f_{11}(w_2^1) = -(23 - 29i)\sqrt{2} + i((23 + 29i)\sqrt{2}) = (-52 + 52i)\sqrt(2) \\
f_1(w_4^2) &= f_{10}(w_2^0) + w_4^2 \cdot f_{11}(w_2^0) = 2 - 10i - 1(2 - 10i) = 0 \\
f_1(w_4^3) &= f_{10}(w_2^1) + w_4^3 \cdot f_{11}(w_2^1) = -(23 - 29i)\sqrt{2} - i((23 + 29i)\sqrt{2}) = (6 + 6i)\sqrt(2)
\end{aligned}
$$

And finally:

$$
\begin{aligned}
f(w_8^0) &= f_0(w_4^0) + w_8^0 \cdot f_1(w_4^0) = 40 \\
f(w_8^1) &= f_0(w_4^1) + w_8^1 \cdot f_1(w_4^1) = 0 \\
f(w_8^2) &= f_0(w_4^2) + w_8^2 \cdot f_1(w_4^2) = 0 \\
f(w_8^3) &= f_0(w_4^3) + w_8^3 \cdot f_1(w_4^3) = -96 \\
f(w_8^4) &= f_0(w_4^0) - w_8^0 \cdot f_1(w_4^0) = 32 \\
f(w_8^5) &= f_0(w_4^1) - w_8^1 \cdot f_1(w_4^1) = 208 \\
f(w_8^6) &= f_0(w_4^2) - w_8^2 \cdot f_1(w_4^2) = -40 \\
f(w_8^7) &= f_0(w_4^3) - w_8^3 \cdot f_1(w_4^3) = -72
\end{aligned}
$$

As stated in slide 10 of the lecture, one can compute the coefficients by $a_k = 1/8 \cdot f(w_8^{-k})$, so:

$$
\begin{aligned}
a_0 &= 5 \\
a_1 &= -9 \\
a_2 &= -5 \\
a_3 &= 26 \\
a_4 &= 4 \\
a_5 &= -12 \\
a_6 &= 0 \\
a_7 &= 0
\end{aligned}
$$

$\Rightarrow p(x) \cdot q(x) = -12x^5 + 4x^4 + 26x^3 - 5x^2 - 9x + 5.$

Exercise 2: Space Trader's Dilemma *(7 Points)*

A space trader has just landed on a distant planet, known for its rare minerals. As usual, they brought their interstellar cargo pod, which can hold exactly k minerals. There are n types of minerals available on the planet, and each type is available in an unlimited supply. The value of one mineral of type i is a_i, with the values sorted in increasing order.
The trader must carefully select exactly k minerals (potentially taking multiple minerals of the same type). Your task is to find all possible distinct total values that the space trader can achieve by selecting exactly k minerals.

Runtime Complexity

The expected runtime complexity for solving this problem should be $O(W \log W + n \log k)$, where $W = k \cdot a_n$ is the maximum achievable total value using k minerals.

Hint: One approach to solving this problem is to define a polynomial whose coefficients represent possible mineral values, then apply the Discrete Fourier Transform (DFT). After manipulating the resulting point representation, use the inverse transform to obtain the final answer.

Sample Solution

Define the following polynomial: $p(x) = x^{a_1} + \cdots + x^{a_n}$. We wish to calculate $p(x)^k$ because the powers of the polynomial will give the possible total values. We can simply do binary exponentiation and FFT to calculate this polynomial but this will take $O(W \log W \log k)$ time. We can speed up the exponentiation with FFT by using in the point representation $p(\omega)^k$. This will take $O(W \log k)$ because for the complex points we use are $2W - 1$ many. It is in $O(W \log W)$ time.

Exercise 3: Counting k-Inversions in a String *(8 Points)*

You are given a string S consisting of only the characters 'A' and 'B'. For each integer k between 1 and $n-1$ (where n is the length of the string), we define a k-inversion as a pair of indices (i, j) such that:

$$1 \leq i < j \leq n,$$

$$j - i = k,$$

$$S[i] = \text{'B'}, \quad S[j] = \text{'A'}.$$

In other words, a k-inversion is a pair of indices (i, j) such that the character at position i is 'B', the character at position $i + k$ is 'A'. For each $k \in \{1, 2, \ldots, n-1\}$, your task is to compute the number of k-inversions in the string S.

Input

The input consists of a single string S, where the string consists only of the characters 'A' and 'B'. The length of the string is denoted by n.

Output

Output $n-1$ integers. The k-th integer (for $k \in \{1, 2, \dots, n-1\}$) should represent the number of k-inversions in the string.

Example

Input:

```
BABA
```

Output:

```
2
0
1
```

Explanation

Consider the string $S =$ 'BABA', which has length $n = 4$.
- For $k = 1$, the valid pairs are $(1, 2)$ and $(3, 4)$, so there are 2 k-inversions. - For $k = 2$, there are no valid pairs, so the number of k-inversions is 0. - For $k = 3$, the valid pair is $(1, 4)$, so there is 1 k-inversion.

Runtime Complexity

A naive solution would involve iterating through all pairs of indices, leading to a time complexity of $O(n^2)$. We can improve the time complexity to efficiently solve the problem in $O(n \log n)$.

Hint: Try defining two polynomials. The product of these polynomials will reveal coefficients that correspond to counts of k-inversions for specific k values.

Sample Solution

Define two polynomials $A(x) = \sum_{i \in \{S[i]='A'\}} x^i$, $B(x) = \sum_{j \in \{S[j]='B'\}} x^{n-j}$. The answer for k will be given by the coefficient of x^{n+k} because let B on index j then A is at $j+k$ the value of the x exponent will be $n - j + j + k = n + k$ so j position does not matter only A's and B's distance;

Exercise 4: Problem for the exercise session *(0 Points)*

In the Kingdom of *Arithmia*, the Queen's Guards are known for their distinctive shields, each decorated with unique patterns. Recently, the kingdom has faced disturbances, prompting the Queen to assemble a team of Guards, each equipped with a shield, to patrol and protect the land. To be prepared for future challenges, she wishes to calculate the number of possible combinations of shield collections that could be assembled from her Guard squad.
There are n Guards, each carrying a shield adorned in one of m distinct patterns. The Queen desires to select exactly k shields, and she is interested in determining the number of unique combinations of shield patterns that can be chosen.
Guards carrying shields of the same pattern are considered indistinguishable; only the variety of shield patterns within a collection determines its uniqueness. Specifically, two collections are considered different if and only if the counts of shields in each pattern differ between the two. We aim to count all possible subsets of Guards, not just consecutive ones in the lineup.

Your task is to calculate the number of distinct collections of k shields that can be formed and return the answer modulo 1009.

Input

The input consists of the following:

- Contains three integers n, m, and k
- and contains n integers in the range $\{1, 2, \ldots, m\}$, representing the shield patterns of subsequent Guards.

Output

Output a single integer: the number of unique collections of k shields.

Example

Input

```
4 3 2
1 2 3 2
```

Output

```
4
```

Explanation

In this example, there are 4 possible unique subsets of $k = 2$ shields that can be chosen based on the patterns:

$$(1, 2), (1, 3), (2, 2), (2, 3)$$

Thus, the answer is 4.
Running time should be: $O((k \cdot m) \log (k \cdot m))$.