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Algorithm Theory
Sample Solution Exercise Sheet 13

Due: Friday, 31st of January, 2025, 10:00 am

Exercise 1: Generalized Contraction *(5 Points)*

Let $G = (V, E, w)$ be a weighted graph s.t. $w : E \to \mathbb{R}^+$. A cut (A, B) is a partition of V such that $V = A \cup B$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$. We define the *weight of the cut* (A, B) to be the total edge weight crossing the cut.
Devise an algorithm that runs in $O(n^4 \log n)$ rounds and returns a minimum weighted cut *w.h.p.* Argue its correcntess and running time.

Sample Solution

Let $C \subseteq E$, define $w(C) := \sum_{e \in C} w(e)$. Let C be a cut in G, we will abuse the notation and define the weight of the cut to be $w(C) := \sum_{e \in C} w(e)$, (i.e. as if we are considering cut C to be the set of edges of the actual cut).
We will run the same contraction algorithm from the lecture but with a (generalized) sampling probability that takes the weights of the edges into account i.e. the probability of contracting edge e in graph $G = (V, E)$ will be $\frac{w(e)}{\sum_{e \in E} w(e)}$. Now for the correctness, we can prove the same theorem as in the lecture i.e. the probability that this new generalized contraction algorithm outputs a specific minimum cut is at least $\frac{2}{n(n-1)}$. And from there on all the correctness analysis is the same as in the lecture. Indeed, the whole analysis of that of the lecture can be also generalized to consider this new sampling probability and eventually prove that same theorem, the only point to notice is that if the weight of the minimum weight cut of a weighted multigraph G (no self-loops) is k, then for every node the total weight of all edges incident to that node must be at least k.

Finally for the running time, everything is the same as well as the implementation step of sampling a random edge which will also take $O(n^2)$, but one has to pay attention to use weights instead of degrees there (i.e. one has to generalize this implementation step also).

Note (someone had this remark in the tutorial): yes, for edge weight values in $\mathbb{N}$, we can reduce the problem of finding a minimum weighted cut on simple edge wighted graphs to the problem of finding the minimum cut on (unweighted) mutigraphs. Indeed, given a simple graph G with edge weights in $\mathbb{N}$, we construct a new graph G' with the same node set and for the edges we allow multiple edges s.t. each edge $e = \{u, v\}$ in G is now split into $w(e) \in \mathbb{N}$ many edges connecting u and v in G'. Now, notice that if C is a min cut on the multigraph graph G' (which is constructed from G), then C is also a min weighted cut in G. Else, there would exist a cut C' in G s.t. $w(C') < w(C)$, and thus in G' this C' will be also be a cut of size smaller than that of C, which is a contradiction. Thus to find the minimum weighted cut of any weighted simple graph G, it is enough to run the same contraction algorithm in the lecture on its corresponding mutigraph G' (as defined above) and find its minimum cut.

Exercise 2: Modified Contraction *(6 Points)*

Let's modify the contraction algorithm from the lecture in the following way: Instead of contracting a uniform random edge, we choose a uniform random pair of remaining nodes in each step and merge them. That is, as long as there are more than two nodes remaining, we choose two nodes $u \neq v$ uniformly at random and replace them by a new node w. For all edges $\{u, x\}$ and $\{v, x\}$ we add an edge $\{w, x\}$ and remove self-loops created at w.

(a) Give an example graph of size at least n where the above algorithm does not work well, that is, where the probability of finding a minimum cut is exponentially small in n (show that in the second part). *(2 Points)*

(b) Show that for your example the modified contraction algorithm has probability of finding a minimum cut at most a^n for some constant $a < 1$. *(4 Points)*

Sample Solution

(a,b) This algorithm is not efficient. Let (A, B) be a minimum cut. For the edge contraction algorithm we know that it outputs (A, B) if and only if it never contracts an edge crossing (A, B) (chapter 7, part V, slide 8). If the there are k crossing edges, we know that there are $\Omega(k \cdot n)$ edges in the graph and hence the probability to choose a crossing edge is $O(1/n)$ (in the first contraction step). In contrast, for the "node contraction" algorithm, it holds that it outputs (A, B) if it never contracts a "crossing pair", i.e., a pair of nodes $\{a, b\}$ with $a \in A$, $b \in B$, regardless whether there is an edge between a and b. The total number of node pairs is $\binom{n}{2} = \Omega(n^2)$, but the number of crossing pairs can be $\Omega(n^2)$ as well, leading to a constant probability that a crossing pair is chosen.

To formalize this argument, consider the following graph: Let n be even. There are two cliques (= graph with an edge between each pair of nodes) of size $n/2$ and a single edge between these cliques, i.e., an edge $\{u, v\}$ such that u is in the one clique and v in the other, and no more edges between the cliques exist.

So there is a unique minimum cut and we show that the probability that the node contraction algorithm chooses this cut is exponentially small. We only consider the first $n/5$ rounds. In these rounds, there are at least $4n/5$ nodes in the graph, i.e., there are $\binom{4n/5}{2}$ pair of nodes. In order for the minimum cut to survive, the two chosen nodes must be within the same clique. Each clique has size at most $\binom{n/2}{2}$ nodes, i.e., there are at most $2\binom{n/2}{2}$ pairs for which the minimum cut would survive. This yields a probability of at most

$$\frac{2\binom{n/2}{2}}{\binom{4n/5}{2}} = \frac{2\frac{n}{2}\left(\frac{n}{2}-1\right)}{\frac{4n}{5}\left(\frac{4n}{5}-1\right)} = \frac{5}{4}\frac{\left(\frac{n}{2}-1\right)}{\left(\frac{4n}{5}-1\right)} = \frac{\frac{5n}{2}-5}{\frac{16n}{5}-4} \overset{(*)}{<} \frac{\frac{5n}{2}}{\frac{15n}{5}} = \frac{5}{6}\ .$$

$(*)$: For $n > 20$ we have $\frac{n}{5} > 4$ and hence $\frac{16n}{5} - 4 > \frac{15n}{5}$.

It follows that the probability that the minimum cut survives the first $n/5$ rounds is less than $\left(\frac{5}{6}\right)^{n/5} = a^n$ with $a = \left(\frac{5}{6}\right)^{1/5} < 1$, i.e., exponentially small.

Exercise 3: Graph Connectivity *(9 Points)*

Let $G = (V, E)$ be a graph with n nodes and edge connectivity[1] $\lambda \geq \frac{16 \ln n}{\varepsilon^2}$ (where $0 < \varepsilon < 1$). Now every edge of G is removed with probability $\frac{1}{2}$. We want to show that the resulting graph $G' = (V, E')$ has connectivity $\lambda' \geq \frac{\lambda}{2}(1-\varepsilon)$ with probability at least $1 - \frac{1}{n}$. This exercise will guide you to this result.

Remark: If you don't succeed in a step you can use the result as a black box for the next step.

[1]The connectivity of a graph is the size of the smallest cut $(S, V \setminus S)$ in G.

(a) Assume you have a cut of G with size $k \geq \lambda$. Show that the probability that the same cut in G' has size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ is at most $e^{-\frac{\varepsilon^2 k}{4}}$. *(2 Points)*

(b) Let $k \geq \lambda$ be fixed. Show that the probability that at least one cut of G with size k becomes a cut of size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ in G' is at most $e^{-\frac{\varepsilon^2 k}{8}}$.

Hint: You can use that for every $\alpha \geq 1$, the number of cuts of size at most $\alpha\lambda$ is at most $n^{2\alpha}$. (3 Points)

(c) Show that for large n the probability that at least one cut of G with *any* size $k \geq \lambda$ becomes a cut of size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ in G', is at most $\frac{1}{n}$.

Hint: Use another union bound. *(4 Points)*

Sample Solution

(a) Let C be the edges of a cut of size k. For $e \in C$ let $X_e = 1$ if $e \in G'$ and else $X_e = 0$. Let $X = \sum_{e \in C} X_e$. The expectation is $\mathbb{E}[X] = \sum_{e \in C} \mathbb{E}[X_e] = \frac{k}{2}$. We use a Chernoff bound

$$\Pr\big(X < \mathbb{E}[X](1-\varepsilon)\big) \leq \exp\Big(-\frac{\varepsilon^2 \mathbb{E}[X]}{2}\Big) = \exp\Big(-\frac{\varepsilon^2 k}{4}\Big).$$

(b) According to the hint we have at most $n^{\frac{2k}{\lambda}}$ many cuts of size k. For an arbitrary cut of C of size k let $C' := C \cap E'$. With a union bound we obtain

$$\begin{aligned}
\Pr\Big(\bigcup_{\substack{C \text{ cut}\\ |C|=k}} \big(|C'| < \tfrac{k}{2}(1-\varepsilon)\big)\Big) &\leq \sum_{\substack{C \text{ cut}\\ |C|=k}} \Pr\Big(|C'| < \frac{k}{2}(1-\varepsilon)\Big) && \text{(union bound)}\\
&\leq \sum_{\substack{C \text{ cut}\\ |C|=k}} \exp\Big(-\frac{\varepsilon^2 k}{4}\Big) && \text{(a)}\\
&\leq n^{\frac{2k}{\lambda}} \exp\Big(-\frac{\varepsilon^2 k}{4}\Big) && \text{(Hint)}\\
&= \exp\Big(\frac{2k \ln n}{\lambda}\Big) \exp\Big(-\frac{\varepsilon^2 k}{4}\Big) && (\lambda \geq \tfrac{16 \ln n}{\varepsilon^2})\\
&\leq \exp\Big(\frac{\varepsilon^2 k}{8} - \frac{\varepsilon^2 k}{4}\Big) = \exp\Big(-\frac{\varepsilon^2 k}{8}\Big).
\end{aligned}$$

(c) For brevity let $\mathcal{E}(k)$ (for $k \geq \lambda$) be the event $\bigcup_{\substack{C \text{ cut}\\ |C|=k}} \big(|C'| < \frac{k}{2}(1-\varepsilon)\big)$ (recall $C' := C \cap G'$). Then a solution with a geometric series which holds for large n

$$\begin{aligned}
\Pr\Big(\bigcup_{k=\lambda}^{O(n^2)} \mathcal{E}(k)\Big) &\leq \Pr\Big(\bigcup_{k=\lambda}^{\infty} \mathcal{E}(k)\Big) \leq \sum_{k=\lambda}^{\infty} \Pr\big(\mathcal{E}(k)\big) \overset{(b)}{\leq} \sum_{k=\lambda}^{\infty} \exp\Big(-\frac{\varepsilon^2 k}{8}\Big) && \text{(geometric series)}\\
&= \frac{e^{-\frac{\varepsilon^2 \lambda}{8}}}{1 - e^{-\frac{\varepsilon^2}{8}}} = \frac{n^{-2}}{1 - e^{-\frac{\varepsilon^2}{8}}} = \frac{1}{n} \cdot \underbrace{\frac{1}{n \cdot (1 - e^{-\frac{\varepsilon^2}{8}})}}_{\text{fraction } \leq 1 \text{ for large } n} \leq \frac{1}{n}.
\end{aligned}$$