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Algorithm Theory
Sample Solution Exercise Sheet 14

Due: Friday, 7th of February, 2025, 10:00 am

Exercise 1: Prof. Jot *(10 Points)*

Suppose you are given a list of N integers $L = [a_1, a_2, \dots, a_N]$, a_i are positive numbers, and a positive integer C. The problem is to find a subset $S \subseteq \{1, 2, \dots, N\}$ such that

$$T(S) = \sum_{i \in S} a_i \leq C,$$

and $T(S)$ is as large as possible.

(a)

Prof. Jot proposes the following greedy algorithm for obtaining an approximate solution to this maximization problem:

Algorithm 1 Greedy Algorithm for Bounded Set Sum

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Require: List of integers [a_1, ..., a_N], capacity C
Ensure: A subset S such that T(S) is maximized under the constraint T(S) ≤ C
S ← ∅, T ← 0
for i = 1 to N do
    if T + a_i ≤ C then
        S ← S ∪ {i}
        T ← T + a_i
return S
```

Show that Prof. Jot's algorithm is not a ρ-approximation algorithm for any fixed value ρ. (Use the convention that $\rho > 1$.)

(b)

Describe a 2-approximation algorithm for this maximization problem that runs in $O(N \log N)$ time.

Sample Solution

Solution

Exercise 2: Miscellaneous Approximations *(10 Points)*

Let $G = (V, E)$ be an undirected connected graph. A set $D \subseteq V$ is called a *dominating set* if each node in V is either contained in D or adjacent to a node in D.
We consider the following randomized algorithm for d-regular graphs (i.e., graphs in which each node has exactly d neighbors).

Algorithm 2 `domset`(G)

1: $D \leftarrow \emptyset$
2: Each node joins D independently with probability $p \leftarrow \min\{1, \frac{c \cdot \ln n}{d+1}\}$ for some constant $c \geq 1$
3: Each node that is neither in D nor has a neighbor in D joins D
4: **return** D

(a) The *minimum dominating set* problem asks to find a dominating set $D \subseteq V$ of minimum size. Show that for $c \geq 2$, the `domset` algorithm computes an $\mathcal{O}(\ln n)$-approximation of a minimum dominating set with probability at least $1 - \frac{2}{n}$. *(3 Points)*

(b) 1. An *independent set* is a set $I \subseteq V$ such that no two nodes in I share an edge in E. The *maximum independent set* problem asks to find an independent set of maximum size. Recall that the *minimum vertex cover* problem asks to find a vertex cover of minimum size. Now, show that both optimization problems are equivalent i.e. finding the minimum-size vertex cover is equivalent to finding the maximum-size independent set . *(2 Points)*

2. Show that the two problems are not equivalent in an approximation-preserving way, i.e it is not true that for all positive integer α, finding an α-approximate minimum vertex cover is equivalent to finding a α-approximate maximum independent set.

Hint: Give a counterexample by finding a family of graphs where one can easily obtain a 2-approximate minimum vertex cover, but this will equivalently find a very bad approximate maximum independent set. *(5 Points)*

Sample Solution

- For every $v \in V$:

$$\begin{aligned}\Pr(v \in D) &= \Pr(v \text{ joins } D \text{ in line 2}) + \Pr(v \text{ joins } D \text{ in line 3}) \\ &= \frac{c \ln n}{d+1} + \left(1 - \frac{c \ln n}{d+1}\right)^{d+1} \\ &\leq \frac{\ln n}{d+1} + e^{-c \ln n} \\ &= \frac{c \ln n}{d+1} + \frac{1}{n^c}.\end{aligned}$$

We obtain:

$$E[|D|] \leq n \cdot \left(\frac{c \ln n}{d+1} + \frac{1}{n^c}\right) = \frac{c \cdot n \ln n}{d+1} + \frac{1}{n^{c-1}} \leq \frac{c \cdot n \ln n}{d+1} + 1.$$

- For each node v, let X_v be a random variable with $X_v = 1$ if v joins D in line 2 and $X_v = 0$ otherwise. Let $X = \sum X_v$. We have $\Pr(X_v = 1) = \frac{c \cdot \ln n}{d+1}$ and hence $\mu = E[X] = \frac{cn \cdot \ln n}{d+1}$.

For $\delta = 3$, we obtain:

$$\Pr(X \geq (1+3)\mu) \leq e^{-\mu} = e^{-\frac{cn \ln n}{d+1}} \leq e^{-c \ln n} = \frac{1}{n^c} \leq \frac{1}{n}.$$

Thus, with probability at least $1 - \frac{1}{n}$, we have $|D| \leq 4\mu = O\left(\frac{n \ln n}{d}\right)$.

- For every $v \in V$,

$$\Pr(v \in D \text{ in line 3}) = (1-p)(1-p)^d \leq e^{-c \ln n} = \frac{1}{n^c}.$$

Moreover,

$$\Pr\left(\bigcup_{v \in V} v \in D \text{ in line 3}\right) \leq \sum_{v \in V} \Pr(v \in D \text{ in line 3}) \leq \frac{n}{n^c} \leq \frac{1}{n}, \quad \text{for } c \geq 2.$$

Thus, with probability at least $1 - \frac{1}{n}$, no node joins D in line 3.

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- In general, let A, B be two events such that $A \subseteq B$. Then, $\Pr(A) \leq \Pr(B)$. Hence,

$$\Pr\left(\texttt{domset} \text{ returns a dominating set of size } O\left(\frac{n \ln n}{d+1}\right) \text{ at the end of its execution}\right)$$

$$\geq \Pr\left(\texttt{domset} \text{ returns a dominating set of size } O\left(\frac{n \ln n}{d+1}\right) \text{ at the end of line 2}\right)$$

- Define the events B_1 and B_2 as follows:
  - B_1: Too many nodes are selected in line 2.
  - B_2: Any additional node is added in line 3.

  By the union bound,

$$\Pr(B_1 \cup B_2) \leq \Pr(B_1) + \Pr(B_2) = \frac{2}{n}.$$

  Hence, the probability that everything proceeds correctly is given by

$$\Pr(B_1^c \cap B_2^c) = 1 - \Pr(B_1 \cup B_2) \geq 1 - \frac{2}{n}.$$

- Notice that each node in a minimum dominating set covers at most $d+1$ nodes. Thus, one can deduce that $\text{OPT} \cdot (d+1) \geq n$, where OPT is the size of a minimum dominating set. This is sufficient to establish the desired result.

1. One can prove that the statement is true by taking the complement of the result, i.e., the set of vertices is a vertex cover if and only if its complement is an independent set.
2. Consider a complete bipartite graph $K_{n,n}$. One can show that, on one hand, all the nodes make up a 2-approximation to the minimum vertex cover problem. However, the complement graph, which is the empty graph of size 0, is far from being a 2-approximation of the maximum independent set problem. (One can show that a maximum independent set is of size n.)