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Algorithm Theory
Sample Solution Exercise Sheet 15

Due: Wednesday, 14th of February 2025, 11:59 pm

Exercise 1: Ticket Problem *(10 Bonus Points)*

A student from Freiburg is doing a one-year internship in Berlin, hence he will have to commute between the two cities. A train ticket from Freiburg to Berlin as well as from Berlin to Freiburg costs $p_0 > 0$ Euros. However, to save money there is a special ticket called *'RailCard50'* that is valid for the whole year and allows buying train tickets for half of the price. The *RailCard50* itself costs $p_1 = 10 \cdot p_0$ Euros. Consider this problem as an online problem, where the number of train rides $x \geq 1$ between these cities during the year is not known beforehand. So before each trip, if not bought yet, the student must make a decision on whether or not to buy the *RailCard50*.

(a) Describe the best offline strategy OPT (x is known beforehand) and give the costs as function depending on x. *(2 Points)*

(b) Assume the student decides on the online strategy ALG_1 (x is unknown), that is to buy the RailCard50 before the first train ride. Give an upper bound on the strict competitive ratio of ALG_1. *(3 Points)*

(c) Give an online strategy ALG_2 that is strictly $\frac{3}{2}$-competitive and prove it. *(5 Points)*

Sample Solution

(a) The best offline strategy is either buying the RailCard50 before the first ride or never buying a RailCard50. Hence,

$$OPT = \min\left\{\ x \cdot p_0, p_1 + x \cdot \frac{p_0}{2}\right\} = p_0 \cdot \min\left\{\ x, 10 + \frac{x}{2}\right\}$$
$$= p_0 \cdot \begin{cases} x & x \leq 20 \\ 10 + x/2 & x > 20 \end{cases}$$

(b) Buying the RailCard50 on the first day, leads to the following cost $ALG_1 = p_1 + x \cdot p_0/2 = p_0(10 + x/2)$. Since we have $OPT \geq p_0 \cdot x$, we get

$$\frac{ALG_1}{OPT} \leq \frac{p_0(10 + x/2)}{p_0 x} \leq \frac{10}{x} + \frac{1}{2} \leq 11$$

(c) The online strategy ALG_2 works as follows: For the first 20 rides, the student does not buy the RailCard50. If there is a 21st ride, the student will buy the RailCard50. We thus have

- If $x \leq 20$: $ALG_2 = p_0 \cdot x = OPT$.
- If $x > 20$: $ALG_2 = 20 \cdot p_0 + p_1 + (x - 20) \cdot p_0/2 = p_0 \cdot (20 + x/2)$, thus:

$$\frac{ALG_2}{OPT} = \frac{p_0 \cdot (20 + x/2)}{p_0 \cdot (10 + x/2)} = \frac{40 + x}{20 + x} = 1 + \frac{20}{20 + x} \leq 1 + \frac{20}{40} = \frac{3}{2}$$

Combining both cases leads to the desired competitive ratio.

Exercise 2: Online Bin Packing *(10 Bonus Points)*

The *Online Bin Packing* problem is a variant of the *Knapsack* problem. Here we are given an unlimited number of bins, each with capacity 1. We get a sequence of items $x_1, x_2, ...$, in online fashion and are required to place them into the bins as we receive them (once placed we are not allowed to put an item into another bin). Each item x_i comes with an individual weight $0 < w_i \leq 1$. The goal is to minimize the number of used bins under the constraint that the sum of the weights of the items in one bin do not exceed its capacity.
In this task we consider the **First-Fit (FF)** online strategy: FF fixes the order of bins arbitrarily w.l.o.g. say $b_1, b_2, ...$, and places each item into the first bin (i.e., the bin with the smallest index) that has enough capacity left to hold the item.

(a) Show that FF is strictly 2-competitive. *(7 Points)*
Hint: Let C_i be the total weight of items in bin b_i. First show that for any given pair of bins b_i and b_j with $1 \leq i < j$ containing at least one element it is true that $C_i + C_j > 1$.

(b) Give a sequence of items for which the strictly competitive ratio of FF is no better than $\frac{3}{2}$. *(3 Points)*

Sample Solution

(a) Let C_i be the total weight of items in bin b_i. We now show the helpful statement of the hint is true, i.e., $\forall\ 1 \leq i < j$ we have $C_i + C_j > 1$. For contradiction assume this is not true. Then there is a pair $i < j$ such that $C_i + C_j \leq 1$. Let x be the last item added to b_j with weight w_x. Due to construction, x didn't fit into C_i and hence $C_i > 1 - w_x$ and as x was put into b_j we clearly have $C_j \geq w_x$. Thus, combining these two we get $C_i + C_j > (1 - w_x) + w_x = 1$. Contradiction.
We are now ready to show the actual statement:
Assume FF uses m bins and for each pair of bins (b_i, b_j) with $i \neq j$ we have $C_i + C_j > 1$ (from the hint). Thus, we can create $\lfloor m/2 \rfloor$ disjoint pairs of bins with a total weight of $> \lfloor m/2 \rfloor$. Since even the optimal solution can not do better than filling each bucket with weight 1, we have that $OPT > \lfloor m/2 \rfloor$ and as OPT has to be an integer we can also say $OPT \geq \lceil m/2 \rceil$. The statement of the task follows by $FF/OPT \leq m/\lceil m/2 \rceil \leq 2$.

Alternative Proof: W.l.o.g. we say that FF uses $m > 1$ bins (If $m = 1$ the algorithm is anyway optimal). From the hint we have

$$\sum_{k=1}^{m} \sum_{\substack{i=1 \\ i \neq k}}^{m} (C_i + C_k) > \sum_{k=1}^{m} \sum_{\substack{i=1 \\ i \neq k}}^{m} 1 = m(m-1).$$

Also note that the optimal solution will take $OPT \geq \sum_{i=1}^{m} C_i$ many bins. Thus,

$$OPT \geq \sum_{i=1}^{m} C_i = \frac{1}{2(m-1)} \sum_{k=1}^{m} \sum_{\substack{i=1 \\ i \neq k}}^{m} (C_i + C_k) > \frac{m(m-1)}{2(m-1)} = \frac{m}{2} = \frac{FF}{2}$$

It therefore follows a 2-competitive ratio.

(b) Consider an instance where $4n$ items arrive in online fashion. The first $2n$ items have weight 0.4 and the last $2n$ items have weight 0.6. Clearly the optimal solution will use $OPT = 2n$ bins. FF on the other side will put the first $2n$ items in n bins. Since the remaining $2n$ items do not fit in the previous used bins, FF needs an additional $2n$ bins for them. Hence, for this instance we have $FF/OPT = 3/2$.