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Algorithms Theory, Winter Term 07/08
Assignment 3

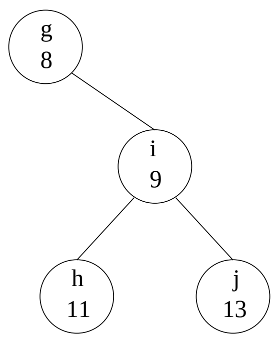
hand in by Monday, December 3, 2007, 14 p.m.
(boxes in building 051)

Exercise 1: Operations on treaps (5 points)

a) Sequentially insert the keys c, d, e, b, a with respective priorities $7, 3, 10, 1, 4$ into an initially empty treap. For all intermediate stages, e.g after performing a rotation, illustrate the state of the treap and specify the operation that leads to this state.

b) Delete the root of the treap resulting from a). Again, illustrate the treap prior to and after each rotation.

c) Merge the treap resulting from b) and the treap shown below. Illustrate all intermediate stages.



Exercise 2: Expected number of rotations (5 points **+ 3 extra points**)

The *left spine* of a binary search tree T is the path from the root to the node with the smallest key. In other words, the left spine is the path from the root that consists of only left edges. Symmetrically, the *right spine* of T is the path from the root consisting of only right edges. The *length* of a spine is the number of nodes it contains (including the root).

a) Consider a treap T immediately after inserting node x. Let C be the length of the right spine of the left subtree of x. Let D be the length of the left spine of the right subtree of x. Argue that the total number of rotations that were performed during the insertion of x is equal to $C + D$.

b) Now, determine the expected values of C and D. Assume that the keys are $1, 2, \ldots, n$. For nodes x and y, $x \neq y$, let $k = \text{key}(x)$ and $i = \text{key}(y)$. Define the indicator random variable

$$X_{i,k} = \begin{cases} 1, & y \text{ is in the right spine of the left subtree of } x \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that $X_{i,k} = 1$ if and only if $\mathrm{prio}(x) < \mathrm{prio}(y)$, $\mathrm{key}(x) > \mathrm{key}(y)$, and for every z such that $\mathrm{key}(y) < \mathrm{key}(z) < \mathrm{key}(x)$, there is $\mathrm{prio}(z) > \mathrm{prio}(y)$.

The following subtasks give 3 extra points:

(ii) Show that $\Pr\{X_{i,k} = 1\} = \frac{(k-i-1)!}{(k-i+1)!} = \frac{1}{(k-i)(k-i+1)}$.

Hint: Consider permuations $\pi = (\pi_1, \ldots, \pi_n)$ of the keys such that π_i is the key whose corresponding priority is the i-th least among all priorities.

(iii) Without proving equality $(*)$ show that $\mathbb{E}[C] = \sum_{j=1}^{k-1} \frac{1}{j(j+1)} \stackrel{(*)}{=} 1 - \frac{1}{k}$.

(iv) Use a symmetry argument to show that $\mathbb{E}[D] = 1 - \frac{1}{n-k+1}$.

(v) Conclude that the expected number of rotations performed when inserting a node into a treap is less than 2.

Exercise 3: Universal hashing (5 points)
Let $N, m \in \mathbb{N}$, $U = [0..N-1]$ and $V = [0..m-1]$, where $[0..i]$ is defined as $[0;i] \cap \mathbb{Z}$ for $i \in \mathbb{N}$. A class $H \subseteq \{h \mid h : U \to V\}$ of functions is *c-universal*, if for all $x, y \in U,\ x \neq y:$

$$\frac{|\{h \in H : h(x) = h(y)\}|}{|H|} \leq \frac{c}{m} .$$

Show that for a c-universal class H of functions, there is $c \geq 1 - \frac{m}{N}$.

Hint: Show that $\sum_{x,y \in U} \delta_h(x,y) \geq N(\frac{N}{m} - 1)$ for each function h (here, Jensen's inequality might be helpful). Then derive that there must exist $x_0, y_0 \in U, x_0 \neq y_0$, with $\sum_{h \in H} \delta_h(x_0, y_0) \geq \frac{|H|}{N}(\frac{N}{m} - 1)$.

Jensen's inequality:
If f is a convex function, x_i is in its domain for $i = 1, \ldots n$, and $w_i \geq 0$ with $\sum_{i=1}^n w_i = 1$, then:

$$\sum_{i=1}^{n} w_i f(x_i) \ \geq\ f(\sum_{i=1}^{n} w_i\ x_i).$$

Exercise 4: Perfect hashing (5 points)
Let $U = \{0, \ldots, 36\}$ and $S = \{3, 5, 7, 17, 19, 20, 23, 28, 30, 32, 35\}$.

a) Use the two-level scheme described in the lecture to build a perfect hash function with $k = 4$ ($N = 37,\ n = |S| = 11$). For $i = 0, \ldots, n-1$ determine the values for W_i, b_i, m_i, k_i and h_{k_i}.

b) For each element of S quote the position in the hash table at which it is stored.