Algorithms Theory

15 – Text search (3)

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Text search

Various scenarios:

Dynamic texts

- Text editors
- Symbol manipulators

Static texts

- Literature databases
- Library systems
- Gene databases
- World Wide Web

Text search

Data type **string**:

- array of character
- file of character
- list of character

Operations (let *T*, *P* be of type **string**)

length: length ()

i-**th character** : *T* [i]

concatenation: cat (*T*, *P*) *T.P*

Problem definition

Given:

text $t_1\, t_2 \ldots.\, t_n \quad \in \Sigma^n$

pattern $p_1 p_2 \ldots\, p_m \quad \in \Sigma^m$

Goal:

Find one or all occurrences of the pattern in the text,
i.e. positions i ($0 \leq i \leq n - m$) such that

$$
\begin{aligned}
p_1 &= t_{i+1} \\
p_2 &= t_{i+2} \\
&\vdots \\
p_m &= t_{i+m}
\end{aligned}
$$

Problem definition

$i \quad i+1 \quad\quad i+m$

text: $t_1 \;\; t_2 \;\; \ldots. \quad t_{i+1} \quad \ldots. \quad t_{i+m} \quad \ldots.. \; t_n$

pattern: $\longrightarrow$ $p_1 \quad \ldots. \quad p_m$

Running time:

1. # possible alignments: $n - m + 1$, # pattern positions: m
   → O(n m)

2. At least 1 comparison per m consecutive text positions:
   → $\Omega\,(\,m + n/m\,)$

Naive method

For each possible position $0 \le i \le n - m$, check at most m character pairs. Whenever a mismatch occurs, shift to the next position.

```
textsearchbf  := proc (T : : string, P : : string)
# Input:      text T, pattern P
# Output:   list L of positions i, at which P occurs in T
    n := length (T); m := length (P);
    L := [ ];
    for i from 0 to n - m do
            j := 1;
            while j ≤ m and T [ i + j ] = P [ j  ]
                        do j := j +1 od;
            if j = m +1 then L := [ L [ ] , i ] fi;
    od;
    RETURN (L)
end;
```

Naive method

Running time:

0 0 ... 0 ... 0 ... 0 0 ...

$\xrightarrow{i}$ $\boxed{0 \;\ldots\; 0 \;\ldots\; 0 \; 1}$

Worst case: $\Omega(m\ n)$

In practice, a mismatch usually occurs very early.

→ running time ~ $c\ n$

The Knuth-Morris-Pratt algorithm (KMP)

Let t_i and p_{j+1} be the characters to be compared:

t_1	t_2	...		...		t_i		...		...
			=	= =	=	$\neq$				
			p_1	...	p_j	p_{j+1}		...	p_m	

If, for a certain alignment, the first mismatch occurs for characters t_i and p_{j+1}, then:

- the last j characters compared in T equal the first j characters of P
- $t_i \neq p_{j+1}$

The Knuth-Morris-Pratt algorithm (KMP)

Idea:

Find $j' = next\,[\,j\,] < j$ such that t_i can then be compared to $p_{j'+1}$.

Find greatest $j' < j$ such that $P_{1\ldots j'} = P_{j-j'+1\ldots j}$.

Find the longest prefix of P that is a proper suffix of $P_{1\ldots j}$.

$t_1 \quad t_2 \quad \ldots \qquad \ldots \qquad t_i \qquad \ldots \qquad \ldots$

$= \;\; = \;\; = \;\; = \qquad \neq$

$\boxed{p_1 \quad \ldots \quad p_j \quad p_{j+1} \quad \ldots \quad p_m}$

The Knuth-Morris-Pratt algorithm (KMP)

Example for determining *next* [*j*]:

t_1 t_2 ... 01011 01011 0 ...

01011 01011 1

01011 01011 1

next [*j*] = length of the longest prefix of *P* that is a proper suffix of $P_{1\ ...j}$

The Knuth-Morris-Pratt algorithm (KMP)

⇒ for P = 0101101011, *next* = [0,0,1,2,0,1,2,3,4,5] :

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	0	1	0	1	1
		0							
		0	1						
					0				
					0	1			
					0	1	0		
					0	1	0	1	
					0	1	0	1	1

The Knuth-Morris-Pratt algorithm (KMP)

```
KMP := proc (T : : string, P : : string)
# Input:     text T, pattern P
# Output:  list L of positions i at which P occurs in T
    n := length(T);   m := length(P);
    L :=  [ ];  next := KMPnext(P);
    j  :=  0;
    for i from 1 to n do
            while j > 0 and T [ i ] <> P [ j+1 ] do j := next [ j ] od;
            if T [ i ]  = P [ j+1 ]  then j := j+1 fi;
            if j = m then L := [ L[ ] , i-m ] ;
                            j := next [ j ]
            fi;
     od;
     RETURN (L);
end;
```

The Knuth-Morris-Pratt algorithm (KMP)

Pattern: abrakadabra, *next* = [0,0,0,1,0,1,0,1,2,3,4]

```
a b r a k a d a b r a b r a b a b r a k ...
| | | | | | | | | | |
a b r a k a d a b r a
```

next [11] = 4

```
a b r a k a d a b r a b r a b a b r a k ...
              - - - - ∤
              a b r a k
```

next [4] = 1

The Knuth-Morris-Pratt algorithm (KMP)

```
a b r a k a d a b r a b r a b a b r a k ...
                    - | | | ∤
                    a b r a k
```

next [4] = 1

```
a b r a k a d a b r a b r a b a b r a k ...
                          - | ∤
                          a b r a k
```

next [2] = 0

```
a b r a k a d a b r a b r a b a b r a k ...
                              | | | | |
                              a b r a k
```

The Knuth-Morris-Pratt algorithm (KMP)

Correctness:

$$
\begin{array}{cccccccccc}
t_1 & t_2 & \ldots & & \ldots & & t_i & \ldots & & \ldots \\
 & & & = & = \; = & = & \neq & & & \\
 & & & p_1 & \ldots & p_j & p_{j+1} & \ldots & p_m &
\end{array}
$$

When starting the for-loop:

$P_{1...j} = T_{i-j...i-1}$ and $j \neq m$

if $j = 0$: we are located at the first character of P

if $j \neq 0$: P can be shifted while $j > 0$ and $t_i \neq p_{j+1}$

The Knuth-Morris-Pratt algorithm (KMP)

If $T[i] = P[j+1]$, j and i can be increased (at the end of the loop).

If P has been compared completely ($j = m$), an occurrence of P in T has been found and we can shift to the next position.

The Knuth-Morris-Pratt algorithm (KMP)

Running time:

- the text pointer *i* is never reset
- text pointer *i* and pattern pointer *j* are always incremented together
- always: *next* [*j*] < *j*;
  j can be decreased only as many times as it has been increased

If the *next*-array is known, the KMP algorithm runs in O(*n*) time.

Computation of the *next*-array

next [i] = length of the longest prefix of P that is a proper suffix of $P_{1 \ldots i}$

next [1] = 0

Let *next* [i-1] = j :

$$\boxed{p_1 \quad p_2 \quad \ldots \qquad \ldots \qquad p_i \qquad \ldots} \qquad \ldots$$

$$= \quad = \quad = \quad = \qquad \neq$$

$$\boxed{p_1 \quad \ldots \quad p_j \quad p_{j+1} \quad \ldots \quad p_m}$$

Computation of the *next*-array

Consider two cases:

1) $p_i = p_{j+1} \rightarrow next\,[\,i\,] = j + 1$

2) $p_i \neq p_{j+1} \rightarrow$ replace j by $next\,[\,j\,]$ until $p_i = p_{j+1}$ or $j = 0$
   If $p_i = p_{j+1}$, set $next\,[\,i\,] = j + 1$, otherwise $next\,[\,i\,] = 0$.

Computation of the *next*-array

```
KMPnext := proc (P : : string)
# Input:    pattern P
# Output:  next-array for P
    m := length (P);
    next := array (1.. m);
    next [1] := 0;
    j := 0;
    for i from 2 to m do
        while j > 0 and P [ i ] <> P [ j+1 ]
            do j := next [ j ] od;
        if P [ i ] = P [ j+1 ] then j := j+1 fi;
        next [ i ] := j
    od;
    RETURN (next);
end;
```

Running time of KMP

The KMP algorithmus runs in O(n + m) time.

Can text search be realized even faster?

The Boyer-Moore algorithm (BM)

Idea: For any alignment of the pattern with the text, scan the characters from right to left rather than from left to right.

Example:

```
h e   s a i d   a b r a k a d a b r a   b u t
    ∤
b u t

h e   s a i d   a b r a k a d a b r a   b u t
          ∤
      b u t
```

The Boyer-Moore algorithm (BM)

```
he said abrakadabra but
        ∤
      but

he said abrakadabra but
           ∤
         but

he said abrakadabra but
              ∤
            but
```

The Boyer-Moore algorithm (BM)

```
he  said  abrakadabra  but
                     ∤
                   but

he  said  abrakadabra  but
                      ∤
                    but

he  said  abrakadabra  but
                         ∤
                       but

he  said  abrakadabra  but
                       | | |
                       but
```

Large jumps:
few comparisons

Desired running time:
$O(m + n/m)$

BM: last-occurrence function

For $c \in \Sigma$ and the pattern P let

$\delta[c]$:= index of the right-most occurrence of c in P

$= \max \{j \mid p_j = c\}$

$$= \begin{cases} 0 & \text{if } c \notin P \\ j & \text{if } c = p_j \text{ and } c \neq p_k \text{ for } j < k \leq m \end{cases}$$

What is the cost for computing all δ-values?

Let $|\Sigma| = l$:

BM: last-occurrence function

Let

c = the character causing the mismatch

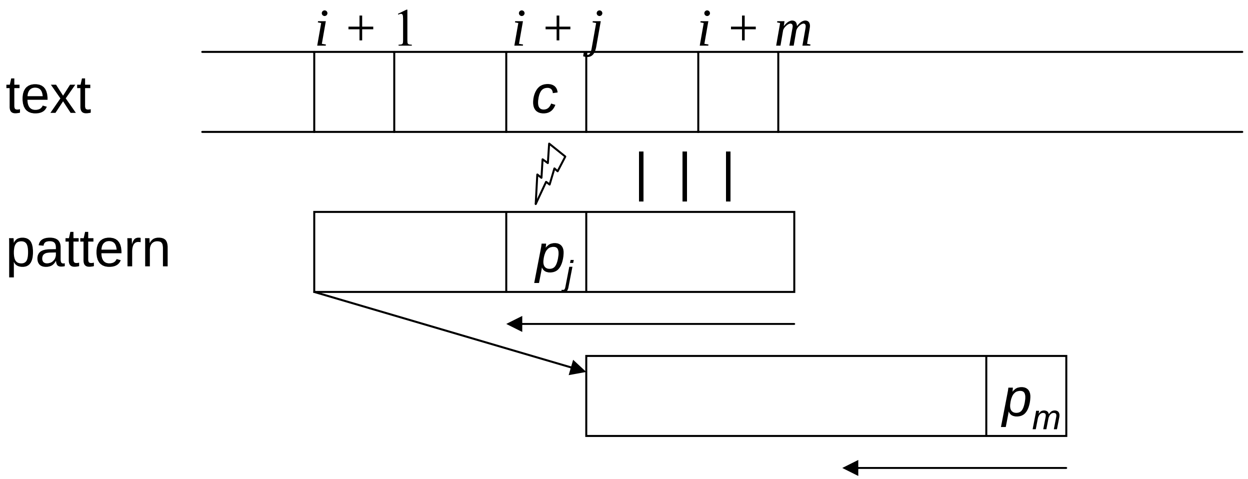
j = the index of the current character in the pattern ($c \neq p_j$)

BM: last-occurrence function

Computation of the pattern shift

Case 1 *c* does not occur in *P* ($\delta[c] = 0$)

Shift the pattern *j* characters to the right.

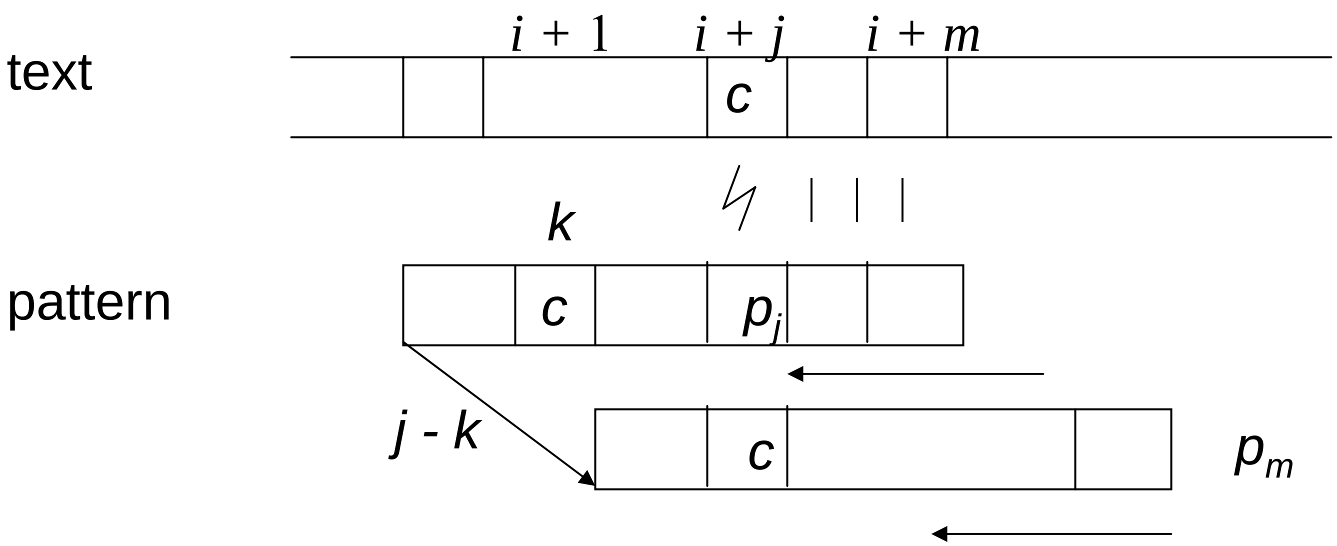


$$\Delta[i] = j$$

BM: last-occurrence function

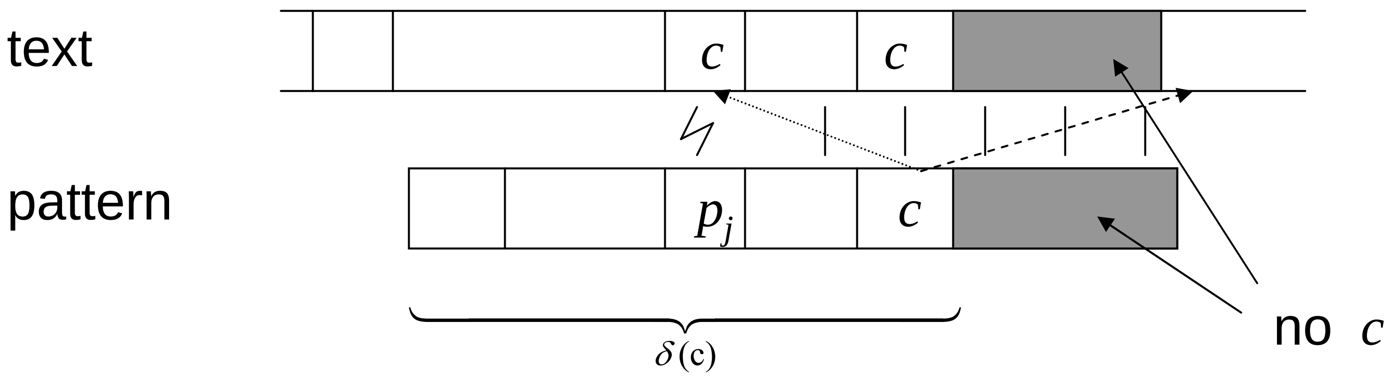
Case 2 *c* occurs in the pattern ($\delta[c] \neq 0$)

Shift the pattern to the right until the rightmost *c* in the pattern is aligned with a potential *c* in the text.



BM: last-occurrence function

Case 2 a: $\delta[c] > j$

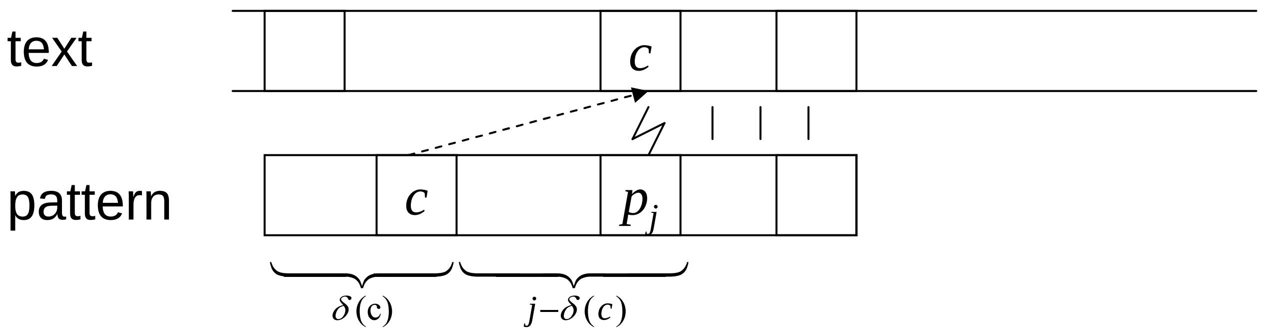


Shift the rightmost *c* in the pattern to a potential *c* in the text.

$$\Rightarrow \text{shift by } \Delta[i] = m - \delta[c] + 1$$

BM: last-occurrence function

Case 2 b: $\delta[c] < j$



Shift the rightmost *c* in the pattern to *c* in the text.

$$\Rightarrow \text{shift by } \Delta[i] = j - \delta[c]$$

BM: Algorithm (version 1)

Algorithm *BM-search1*

Input: text T, pattern P

Output: all positions of P in T

n := length(T); m := length(P)

compute δ

$i := 0$

while $i \leq n - m$ **do**

    $j := m$

    while $j > 0$ **and** $P[j] = T[i + j]$ **do**

        $j := j - 1$

    end while;

BM: Algorithm (version 1)

```
     if j = 0
      then output position i
             i := i + 1
      else if δ[ T[i + j] ] > j
          then i := i + m + 1 - δ[ T[i + j] ]
          else i := i + j - δ[ T[i + j] ]
  end while;
```

BM: Algorithm (version 1)

Analysis:

Desired running time: O($m + n/m$)

Worst-case running time: $\Omega(n\ m)$

0 0 ... 0 0 ... 0 ... 0 ...

$\xrightarrow{i}$ 1 0 ... 0 ... 0

Match heuristic

Use the information collected before a mismatches $p_j \neq t_{i+j}$ occurs.

$t_1 \quad t_2 \quad \dots \qquad t_{i+1} \quad \dots \quad t_{i+j} \quad \dots \quad t_{i+m}$

$\neq \quad = \quad = \quad =$

$i \longrightarrow$ $\boxed{p_1 \quad \dots \quad p_j \quad \dots \quad p_m}$

gsf[j] = position of the end of the next occurrence of the suffix $P_{j+1 \dots m}$ from the right that is not preceded by character P_j (good suffix function)

Possible shift: $\gamma[j] = m - gsf[j]$

Example of computing *gsf*

gsf[*j*] = position of the end of the closest occurrence of the suffix $P_{j+1 \ldots m}$ from the right that is not preceded by character P_j

pattern: banana

gsf[j]	inspected suffix	forbidden character	further occurrence	position
gsf[5]	a	n	banana	2
gsf[4]	na	a	*** bana na	0
gsf[3]	ana	n	banana	4
gsf[2]	nana	a	banana	0
gsf[1]	anana	b	banana	0

Example of computing *gsf*

$\Rightarrow$ *gsf* (banana) = [0,0,0,4,0,2]

```
a b a a b a b a n a n a n a n a
                ≠ = = =
            b a n a n a
                b a n a n a
```

BM: Algorithm (version 2)

Algorithm *BM-search2*

Input: text *T*, pattern *P*

Output: shift for all occurrences of *P* in *T*

```
n := length(T); m := length(P)
compute δ and γ
i := 0
while i ≤ n – m do
    j := m
    while j > 0 and P [ j ] = T [ i + j ] do
        j := j – 1
    end while;
```

BM: Algorithm (version 2)

```
    if j = 0
      then output position i
           i := i + γ [ 0 ]
      else i := i + max( γ [ j ],  j - δ[ T [i + j] ] )
end while;
```