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Sheet 01

Average-Case Analysis

Exercise 1 (John von Neumann's Coin Trick)

Suppose we are given a biased coin that comes up heads with some unknown probability p. Our task is to simulate a fair coin, i.e., one that comes up heads with probability 1/2, given this biased coin only. How can we do that? *Hint.* Look at pairs of tosses of the biased coin.

Exercise 2 (Balls into Bins)

We throw m balls independently and uniformly distributed into n bins.

(1) What is the expected number of bins that remain empty?

(2) How large must m be (in dependency on n) such that with high probability *no* bin remains empty. "With high probability" means with probability tending to one as n tends to infinity.

    Hint. Define random variables X_i that count how many balls are thrown until the number of non-empty bins increases from $i - 1$ to i. Then use Chebyshev's inequality.

Exercise 3 (Bubble Sort)

The algorithm Bubble Sort for sorting an array a of numbers scans the array until it finds an *inversion*, i.e., a position i with $a_i > a_{i+1}$. Then the numbers a_i and a_{i+1} are swapped and the algorithm is restarted until no more inversions exist.
The number of swaps in the worst case is $O\left(n^2\right)$. What is the expected number of swaps if the array is a uniformly drawn permutation of $(1, \ldots, n)$? *Hint.* Define suitable indicator variables and use linearity of expectation.

Exercise 4 (Quickselect)

In the lecture we have seen a proof on the expected number of comparisons of Quickselect using a recursion and induction. Give an alternate proof that uses linearity of expectation similarly to the alternate proof of Quicksort.