Algorithm Theory

06 – Amortized Analysis

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Amortization

- Consider a sequence $a_1, a_2, \ldots, a_n$ of n operations performed on a data structure D

- T_i = execution time of a_i

- $T = T_1 + T_2 + \ldots + T_n$ total execution time

- The execution time of a single operation can vary within a large range, e.g. in $1,\ldots,n$, but the worst case does not occur for all operations of the sequence.

- Average execution time of an operation is small, even though a single operation can have a high execution time.

Analysis of algorithms

- Best case
- Worst case
- Average case
- Amortized worst case

What is the average cost of an operation in a worst case sequence of operations?

Amortization

Idea:

- Pay more for inexpensive operations
- Use the credit to cover the cost of expensive operations

Three methods:

1. Aggregate method
2. Accounting method
3. Potential method

Amortization = Overcharging + Bookkeeping

1. Aggregate method: binary counter

Incrementing a binary counter: determine the bit flip cost

Operation	Counter value	Cost
	00000	
1	00001	1
2	00010	2
3	00011	1
4	00100	3
5	00101	1
6	00110	2
7	00111	
8	01000	
9	01001	
10	01010	
11	01011	
12	01100	
13	01101	

2. The accounting method

Observation:

In each increment exactly one 0 flips to 1.

Idea:

Pay two cost units for flipping a 0 to a 1

→ each 1 has one cost unit deposited in the banking account

The accounting method

Operation	Counter value
	0 0 0 0 0
1	0 0 0 0 1
2	0 0 0 1 0
3	0 0 0 1 1
4	0 0 1 0 0
5	0 0 1 0 1
6	0 0 1 1 0
7	0 0 1 1 1
8	0 1 0 0 0
9	0 1 0 0 1
10	0 1 0 1 0

3. The potential method

Potential function ϕ

Data structure $D \rightarrow \phi(D)$

t_i = actual cost of the i-th operation

ϕ_i = potential after execution of the i-th operation (= $\phi(D_i)$)

a_i = amortized cost of the i-th operation

Definition:

$$a_i = t_i + \phi_i - \phi_{i-1}$$

Example: binary counter

D_i = counter value after the i-th operation
$\phi_i = \phi(D_i)$ = # of 1's in D_i

i–th operation	# of 1's
D_{i-1}: .....0/1.....01.....1	B_{i-1}
D_i : .....0/1.....10.....0	$B_i = B_{i-1} - b_i + 1$

t_i = **actual bit flip cost of operation *i***
= $b_i + 1$

Binary counter

t_i = actual bit flip cost of operation *i*
a_i = amortized bit flip cost of operation *i*

$$
\begin{aligned}
a_i &= (b_i + 1) + (B_{i-1} - b_i + 1) - B_{i-1} \\
&= 2 \\
&\Rightarrow \sum t_i \leq 2n
\end{aligned}
$$

Dynamic tables

Problem:

Maintain a table supporting the operations insert and delete such that

- the table size can be adjusted dynamically to the number of items
- the used space in the table is always at least a constant fraction of the total space
- the total cost of a sequence of n operations (insert or delete) is $O(n)$.

Applications: hash table, heap, stack, etc.

Load factor α_T: number of items stored in the table divided by the size of the table

Implementation of ‘insert’

```
class dynamic table {

    int [] table;

    int size;               // size of the table
    int num;                // number of items

dynamicTable() {            // initialization of an empty table
    table = new int [1];
    size = 1;
    num = 0;
    }
```

Implementation of ‘insert’

```
insert ( int x) {
    if (num == size ) {
        newTable = new int [2*size];
        for (i = 0; i < size; i++)
            insert table[i] into newTable;
        table = newTable;
        size = 2*size;
    }
    insert x into table;
    num = num + 1;
}
```

Cost of *n* insertions into an initially empty table

t_i = cost of the *i*-th insert operation

Worst case:

$t_i = 1$ if the table is not full prior to operation *i*

$t_i = (i - 1) + 1$ if the table is full prior to operation *i*.

Thus *n* insertions incur a total cost of at most

$$\sum_{i=1}^{n} i = \Theta\left(n^2\right)$$

Amortized worst case:

Aggregate method, accounting method, potential method

Potential method

T table with

- $k = T.num$ items
- $s = T.size$ size

Potential function

$$\phi(T) = 2k - s$$

Potential method

Properties

- $\phi_0 = \phi(T_0) = \phi$ (empty table) = -1
- Immediately before a table expansion we have $k = s$, thus $\phi(T) = k = s.$
- Immediately after a table expansion we have $k = s/2$, thus $\phi(T) = 2k - s = 0$.
- For all $i \geq 1$: $\phi_i = \phi(T_i) > 0$
  Since $\phi_n - \phi_0 \geq 0$,

$$\sum t_i \leq \sum a_i$$

Amortized cost a_i of the *i*-th insertion

k_i = # items stored in T after the *i*-th operation

s_i = table size of T after the *i*-th operation

Case 1: *i*-th operation does not trigger an expansion

$$k_i = k_{i-1} + 1, \; s_i = s_{i-1}$$

$$\begin{aligned} a_i &= 1 + (2k_i - s_i) - (2k_{i-1} - s_{i-1}) \\ &= 1 + 2(k_i - k_{i-1}) \\ &= 3 \end{aligned}$$

Case 2: *i*-th operation does trigger an expansion

$k_i = k_{i-1} + 1,\ s_i = 2s_{i-1}$

$a_i = k_{i-1} + 1 + (2k_i - s_i) - (2k_{i-1} – s_{i-1})$

$= 3$

Inserting and deleting items

Now: Contract the table whenever the load becomes too small.

Goal:

(1) The load factor is bounded from below by a constant.

(2) The amortized cost of a table operation is constant.

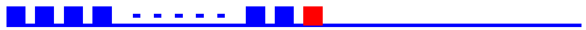
First approach

- Expansion: as before
- Contraction: Halve the table size when a deletion would cause the table to become less than half full.

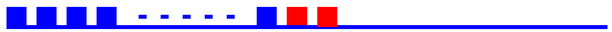
„Bad“ sequence of table operations

Cost

n/2 ‘insert’ op. (table is full)  3 *n/2*

I: expansion  *n/2* + 1

D, D: contraction  *n/2* + 1

I, I : expansion  *n/2* + 1

D, D: contraction 

Total cost of the sequence of operations:
$I_{n/2}$, *I,D,D,I,I,D,D*,... of length *n* is

Second approach

Expansion: Double the table size when an item is inserted into a full table.

Contraction: Halve the table size when a deletion causes the table to become less than ¼ full.

Property: At any time the table is at least ¼ full, i.e.

$$\tfrac{1}{4} \le \alpha(T) \le 1$$

What is the cost of a sequence of table operations?

Analysis of ‘insert’ and ‘delete’ operations

k = *T.num*,   *s* = *T.size*, $\alpha = k/s$

Potential function ϕ

$$\boxed{\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}}$$

Analysis of ‘insert’ and ‘delete’ operations

$$\boxed{\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}}$$

Immediately after a table expansion or contraction:

$s = 2k$, thus $\phi(T) = 0$

Analysis of an ‘insert’ operation

i-th operation: $k_i = k_{i-1} + 1$

Case 1: $\alpha_{i-1} \geq ½$

Case 2: $\alpha_{i-1} < ½$

Case 2.1: $\alpha_i < ½$

Case 2.2: $\alpha_i \geq ½$

Analysis of an ‘insert’ operation

Case 2.1: $\alpha_{i-1} < ½$, $\alpha_i < ½$ no expansion

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

Analysis of an 'insert' operation

Case 2.2: $\alpha_{i-1} < ½$, $\alpha_i \geq ½$ no expansion

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

Analysis of a 'delete' operation

$k_i = k_{i-1} - 1$

Case 1: $\alpha_{i-1} < ½$

Case 1.1: deletion does not trigger a contraction

$s_i = s_{i-1}$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

Analysis of a 'delete' operation

$k_i = k_{i-1} - 1$

Case 1: $\alpha_{i-1} < ½$

Case 1.2: $\alpha_{i-1} < ½$ deletion does trigger a contraction

$s_i = s_{i-1}/2 \quad k_{i-1} = s_{i-1}/4$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$

Analysis of a 'delete' operation

Case 2: $\alpha_{i-1} \geq$ ½ no contraction

$s_i = s_{i-1} \quad k_i = k_{i-1} - 1$

Case 2.1: $\alpha_i \geq$ ½

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$

Analysis of a 'delete' operation

Case 2: $\alpha_{i-1} \geq ½$ no contraction

$$s_i = s_{i-1} \quad k_i = k_{i-1} - 1$$

Case 2.2: $\alpha_i < ½$

Potential function ϕ

$$\phi(T) = \begin{cases} 2k - s, \text{if } \alpha \geq 1/2 \\ s/2 - k, \text{if } \alpha < 1/2 \end{cases}$$