Algorithms Theory

12 – Minimum Spanning Trees

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25.1. : No lecture
1.2. : 90 minute lecture

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1. Minimum spanning trees

$G = (V, E)$ undirected graph $w: E \rightarrow R$ weight function

Let $T \subseteq E$ be a tree (connected, acyclic subgraph).

Total weight of T :

$$w(T) = \sum_{(u,v)\in T} w(u,v)$$

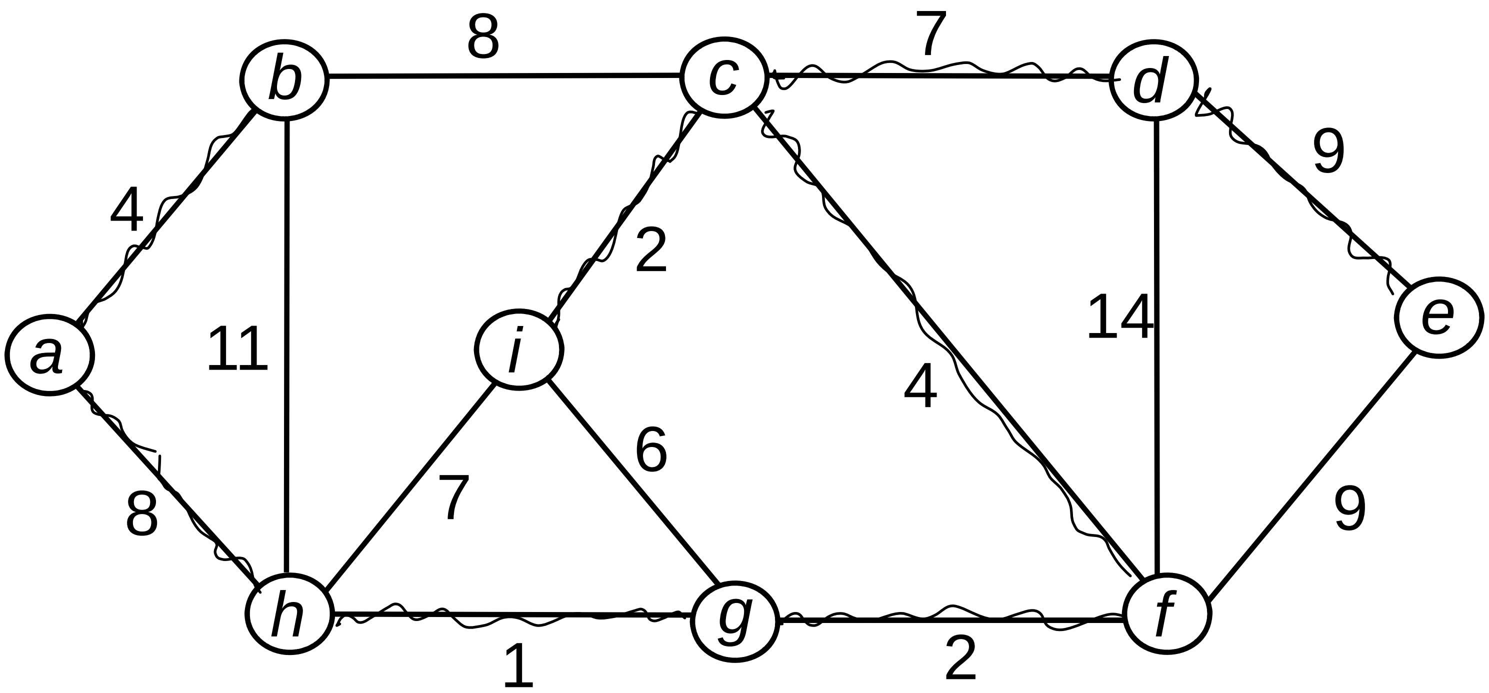
$w(a,b)=4$ $w(b,d)=9$

$T = \{(a,b),(b,d),(d,e),(b,c)\}$

$w(T) = 4 + 9 + 15 + (-3) = 25$

Minimum spanning trees

A tree $T \subseteq E$ that connects all vertices in V and whose total weight is minimal is called a minimum spanning tree.



Growing a minimum spanning tree

Invariant: Maintain a set $A \subseteq E$ that is a subset of some minimum spanning tree.

$A = \emptyset$ initially

$A \subseteq T$ for some MST T.

Definition: An edge $(u,v) \in E \setminus A$ is a safe edge for A if $A \cup \{(u,v)\}$ is also a subset of some minimum spanning tree.

$A \subseteq A \cup \{(u,v)\} \subseteq T$ for some MST T.

(u,v) can be added to A.

Not clear how to find a safe edge for A.

$$\begin{array}{ccccccccc}
T_0 & & T_1 & & T_2 & & & & T_{n-1} \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & & & \rotatebox[origin=c]{90}{$\subseteq$} \\
A_0 & \xrightarrow[e_1]{} & A_1 & \xrightarrow[e_2]{} & A_2 & \longrightarrow & \cdots & & A_{n-1} \\
\| & & & & & & & & \\
\emptyset & & & & & & & &
\end{array}$$

Greedy approach

Algorithm Generic-MST(G,w);

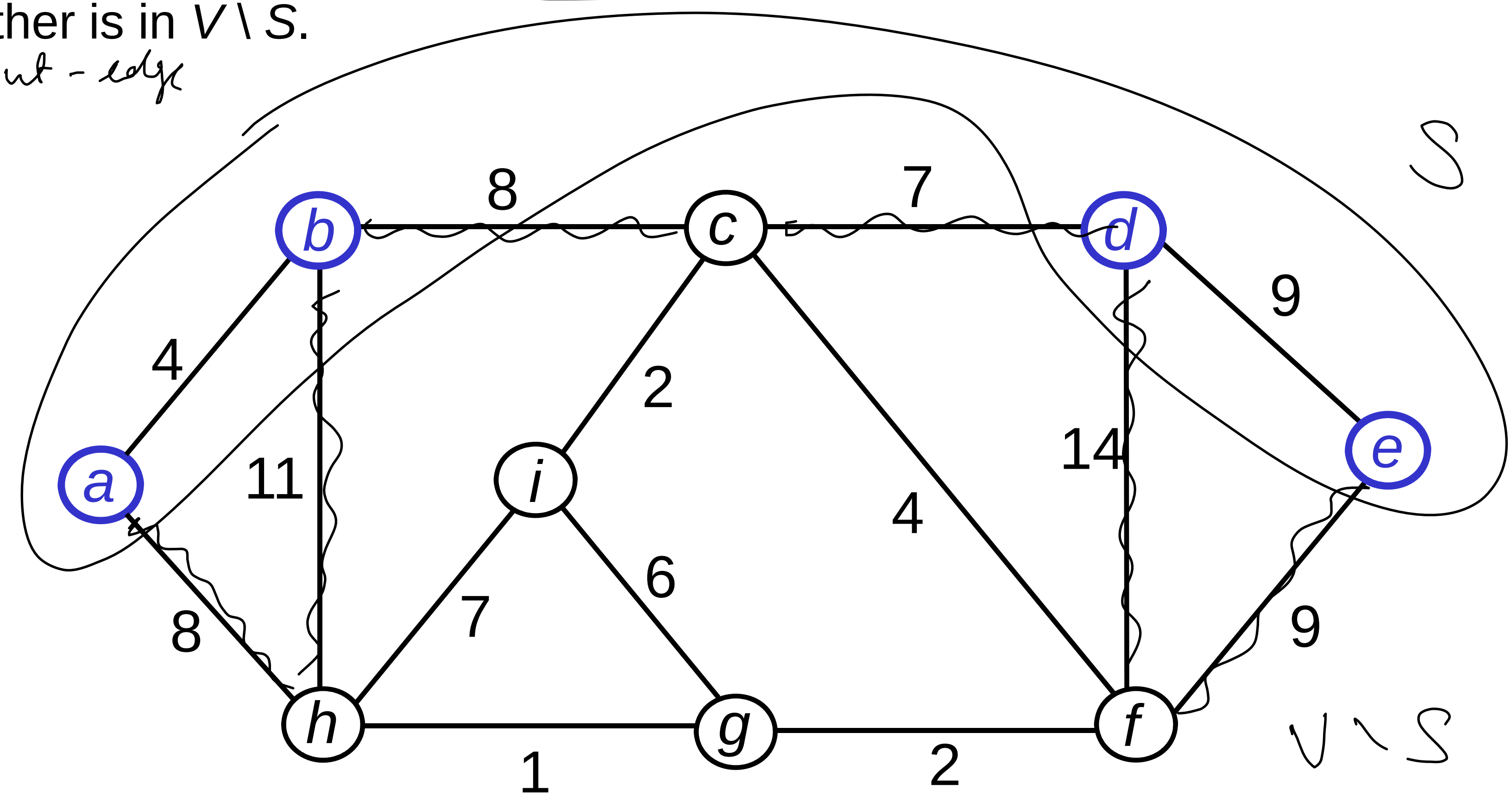
1. $A \leftarrow \emptyset$;
2. **while** A does not form a spanning tree **do**
3. Find an edge (u,v) that is safe for A;
4. $A \leftarrow A \cup \{(u,v)\}$;
5. **endwhile;**

2. Cuts

A cut $(S, V \setminus S)$ is a partition of V. $S \subseteq V$

An edge (u,v) crosses $(S, V \setminus S)$ if one of its endpoints is in S and the other is in $V \setminus S$.

cut-edge

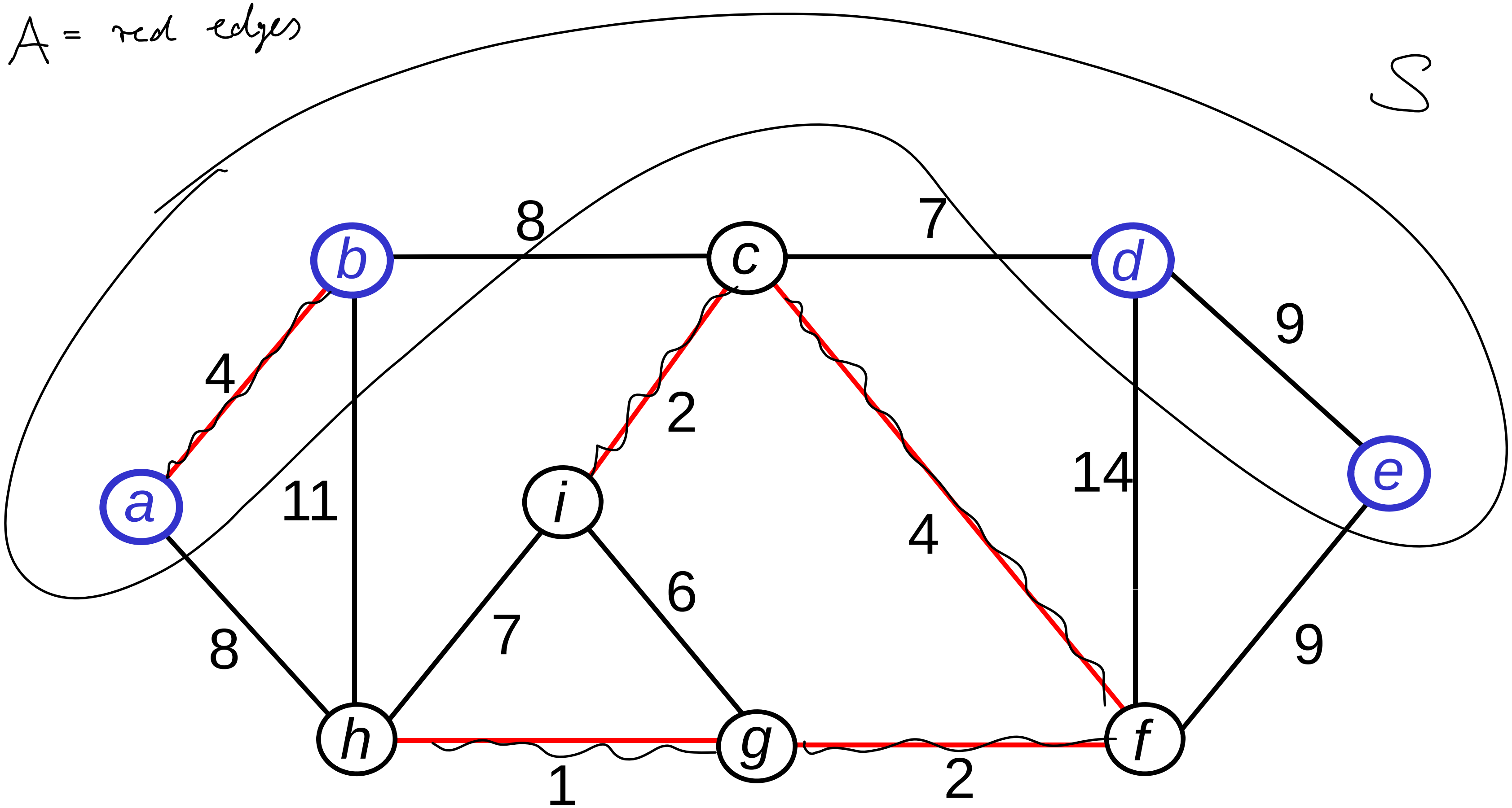


Cuts

Let A be a set of edges and $(S, V \setminus S)$ be a cut.

A cut respects a set *A* of edges if no edge in *A* crosses the cut.

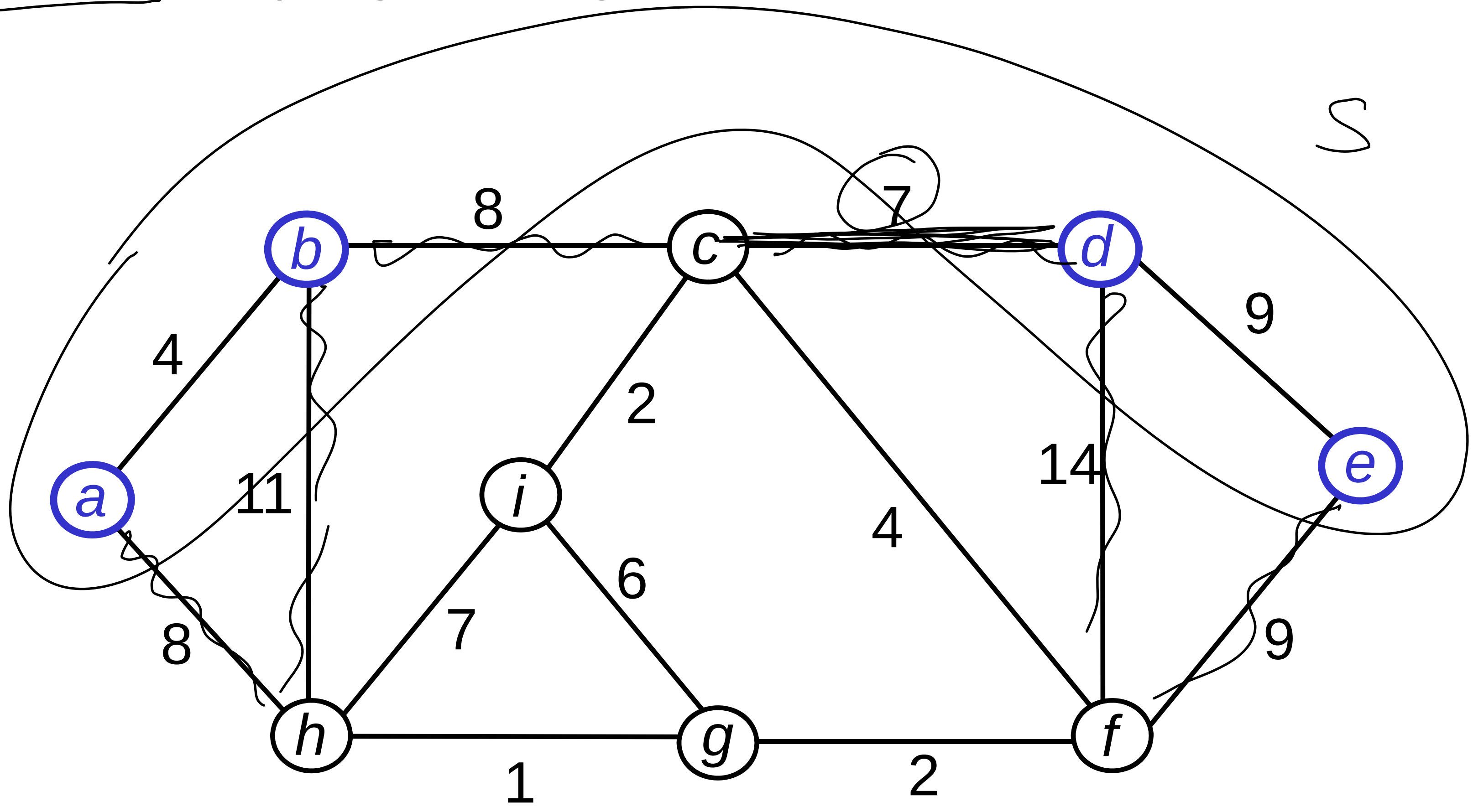
A = red edges



Cuts

Let (S, V \ S) be a cut.

An edge is a light edge crossing a certain cut if its weight is the minimum of any edge crossing the cut.



3. Safe edges

Theorem: Let A be a subset of some minimum spanning tree T, and let $(S, V \setminus S)$ be a cut that respects A. If (u,v) is a light edge crossing $(S, V \setminus S)$ then (u,v) is safe for A.

Proof:

Case 1: $(u,v) \in T$: ok

Case 2: $(u,v) \notin T$:

We construct another minimum spanning tree T' with $(u,v) \in T'$ and $A \subseteq T'$.