Nash Stability in Social Distance Games

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Goal and Motivation

- The aim of this work is to improve our understanding of social networks from the viewpoint of non-cooperative game theory.
- Social Distance Games: a model of interaction on social networks capturing the idea that social networks exhibit homophily (agents prefer to maintain ties with agents who are close to them.).
- Study the Nash equilibria in this context, focusing on the Price of Anarchy (PoA), Price of Stability (PoS) and the convergence into a Nash stable solution.

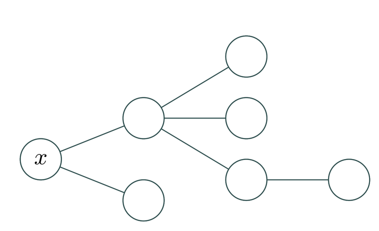
Model: Social Distance Games (SDGs)

A SDG [Brânzei and Larson 2011] is represented as an undirected graph $G = (V, E)$

- V is the set of agents and E is the set of links between agent.
- The *utility* of an agent $x \in V$ in a given coalition C is a suitable function of her harmonic-centrality in the subgraph induced by C, that is:

$$u_x(C) = \frac{1}{|C|} \sum_{y \in C \setminus \{x\}} \frac{1}{d_C(x,y)}.$$

Example



The utility of agent x in this coalition is:

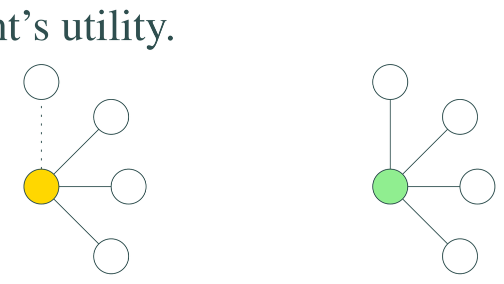
$$u_x = \frac{2 \times 1 + 3 \times \frac{1}{2} + 1 \times \frac{1}{3}}{7} = \frac{23}{42}$$

Properties of SDGs

1. An agent prefers direct connections over indirect ones.

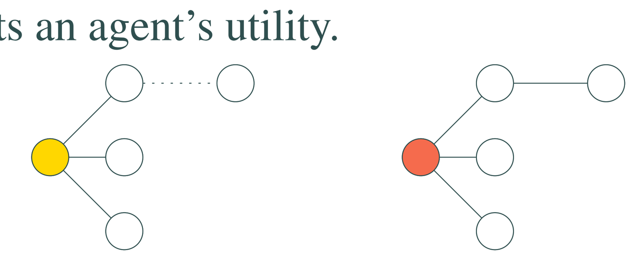
$0.5 \Longrightarrow 0.\overline{6}$

2. Adding a close connection positively affects an agent's utility.



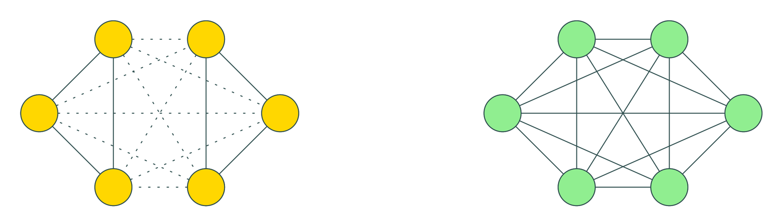
$0.75 \Longrightarrow 0.8$

3. Adding a distant connection negatively affects an agent's utility.



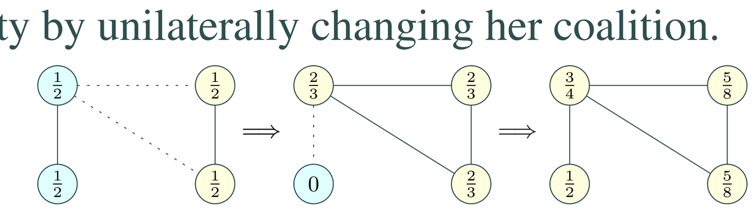
$0.75 \Longrightarrow 0.7$

4. All things being equal, agents prefer larger coalitions.



$0.\overline{6} \Longrightarrow 0.8\overline{3}$

Nash Equilibria Nash stable outcomes are states in which no agent can improve her utility by unilaterally changing her coalition.



- Social Welfare (SW)

$$SW(C) = \sum_{x \in C} u_x$$

- Price of Anarchy (PoA)
  Worst-case ratio

$$\frac{\textit{SW of a best clustering}}{\textit{SW of a Nash stable clustering}}$$

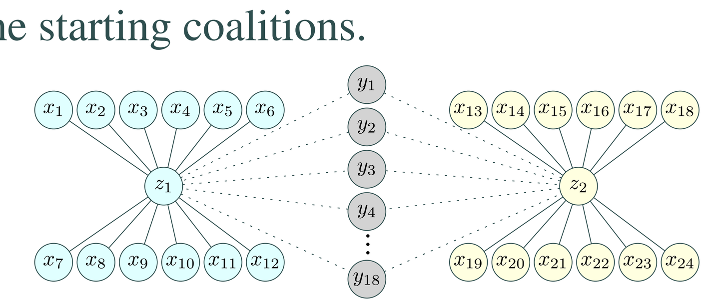
- Price of Stability (PoS)
  Best-case ratio

$$\frac{\textit{SW of a best clustering}}{\textit{SW of a Nash stable clustering}}$$

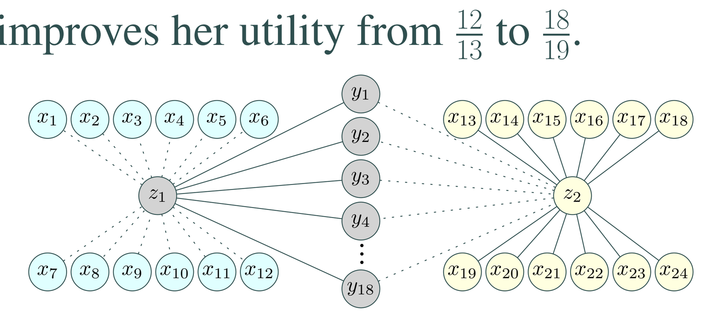
Nash Equilibria: convergence

- SDGs always admit a Nash equilibrium: the grand coalition is Nash stable as no agent can have any improving deviation.
- SDGs may not converge to Nash equilibria.

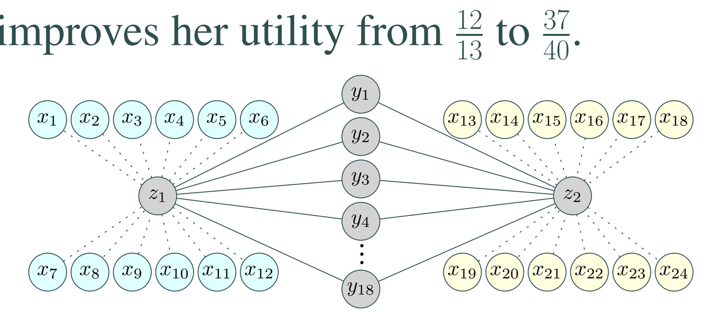
The starting coalitions.



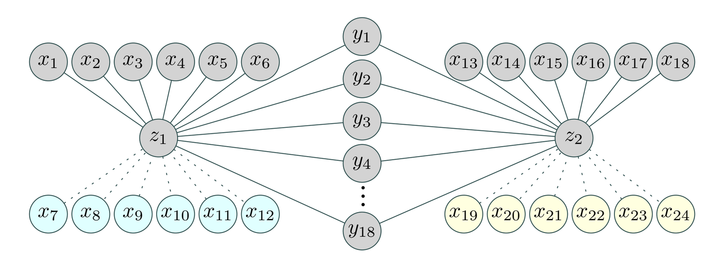
z_1 improves her utility from $\frac{12}{13}$ to $\frac{18}{19}$.



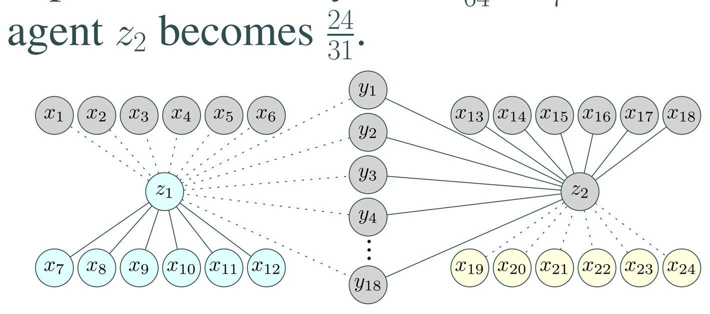
z_1 improves her utility from $\frac{12}{13}$ to $\frac{37}{40}$.



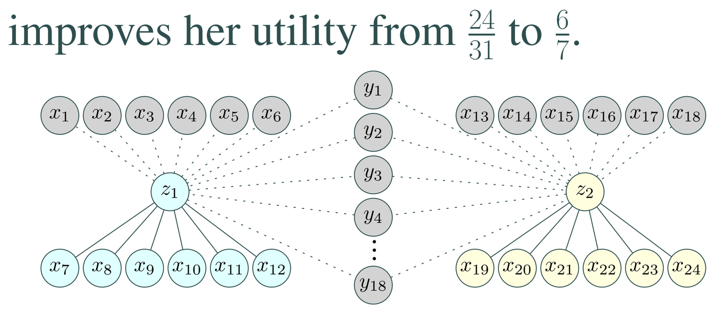
$\{x_1 \dots x_6\}$ and $\{x_{13} \dots x_{18}\}$ have utility 0, so they increase their utility doing the following deviations one after the other, taking the utility of agent z_1 to $\frac{53}{64}$.



z_1 improves her utility from $\frac{53}{64}$ to $\frac{6}{7}$. The utility of agent z_2 becomes $\frac{24}{31}$.



z_2 improves her utility from $\frac{24}{31}$ to $\frac{6}{7}$.



$x_1 \dots x_6$ and $x_{13} \dots x_{18}$ increase their utilities obtaining the initial coalitions.

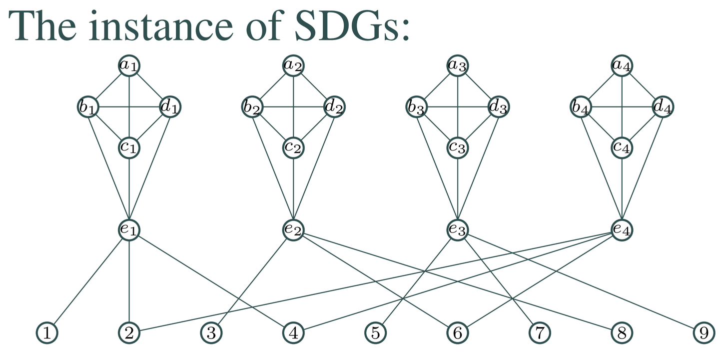
Computing a Best Nash Equilibrium for SDGs is NP-hard

Reduction

- We provide a polynomial time reduction from the NP-Complete RESTRICTED EXACT COVER *by 3-SETS (RXC3)* problem.
- Given a generic instance $(U, \mathcal{B})$ of $RXC3$, with $|U| = 3p$ and $|\mathcal{B}| = m$, we build an instance of SDGs by specifying the underlying undirected graph $G = (V, E)$ as follows:
  - for each triple $B_i \in \mathcal{B}$, for $i \in [m]$, we associate a set of 5 nodes $X_i = \{a_i, b_i, c_i, d_i, e_i\}$.
  - for each element $u_j \in U$, for $j \in [3p]$, we consider a node y_j and a set of 3 edges $E_j = \{(y_j, e_i) | u_j \in B_i\}$.
- Therefore, $|V| = 3p + 5m$ and $E = 9(p + m)$. Clearly such a reduction can be done in polynomial time.

Example of the Reduction

- $\mathcal{B} = \big\{\{1, 2, 4\}, \{3, 6, 8\}, \{5, 7, 9\}, \{2, 4, 6\}\big\}$.
- $U = [1, 9]$.
- The instance of SDGs:



Reduction Result

- If there is an exact cover for the input instance of $RXC3$, then there exists a Nash equilibrium in the reduced instance of SDGs s.t.

$$SW \geq \frac{21}{4}p + \frac{19}{5}(m - p).$$

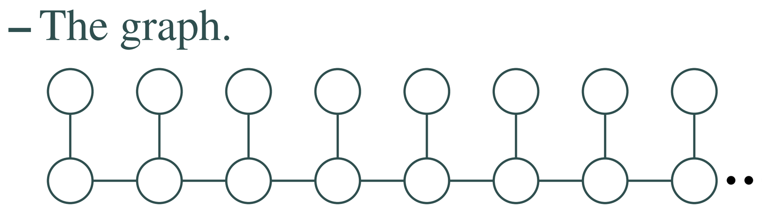
- If there is not an exact cover for the input instance of $RXC3$, then every Nash equilibrium in the reduced instance of SDGs s.t.

$$SW < \frac{21}{4}p + \frac{19}{5}(m - p).$$

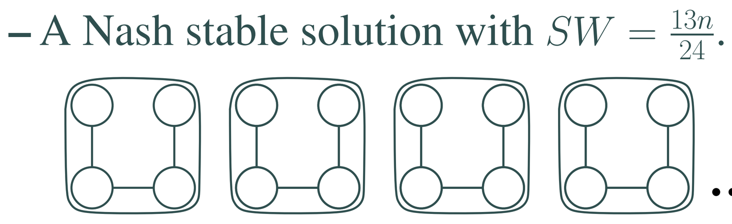
Nash Equilibria: PoA

PoA in SDGs is $\Theta(n)$.

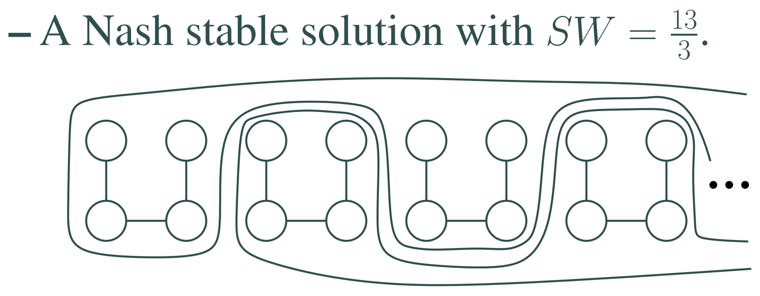
- PoA in SDGs is $O(n)$:
  - the SW is upper bounded by $n - 1$ (grand coalition on complete graphs);
  - in any coalition, the utility of each node is at least $\frac{1}{n}$.
- An SDG with n agents having $\text{PoA} = \Omega(n)$.
  - The graph.



  - A Nash stable solution with $SW = \frac{13n}{24}$.



  - A Nash stable solution with $SW = \frac{13}{3}$.



Nash Equilibria: PoS

- The PoS of SDGs is at least

$$\text{PoS} \geq \frac{6}{5} - \epsilon \; (\forall \epsilon \geq 0).$$

- The PoS of SDGs in which the underlying graph has $girth = 4$ is at least

$$\text{PoS} \geq \frac{169}{160} = 1.05625.$$

- The upper bound of the PoS of SDGs in which the underlying graph has $girth > 4$ (i.e., there are no two agents that have more than one friend in common) is

$$\text{PoS} \leq \frac{1}{2} + \frac{1}{\sqrt{2}} \approx 1.2.$$

Open Problems

- Upper bound of the PoS for general graphs.
- Is there a polynomial time algorithm for determining the existence of a Nash stable clustering for SDGs different from the grand coalition?
- Generalize our results to weighted graphs.