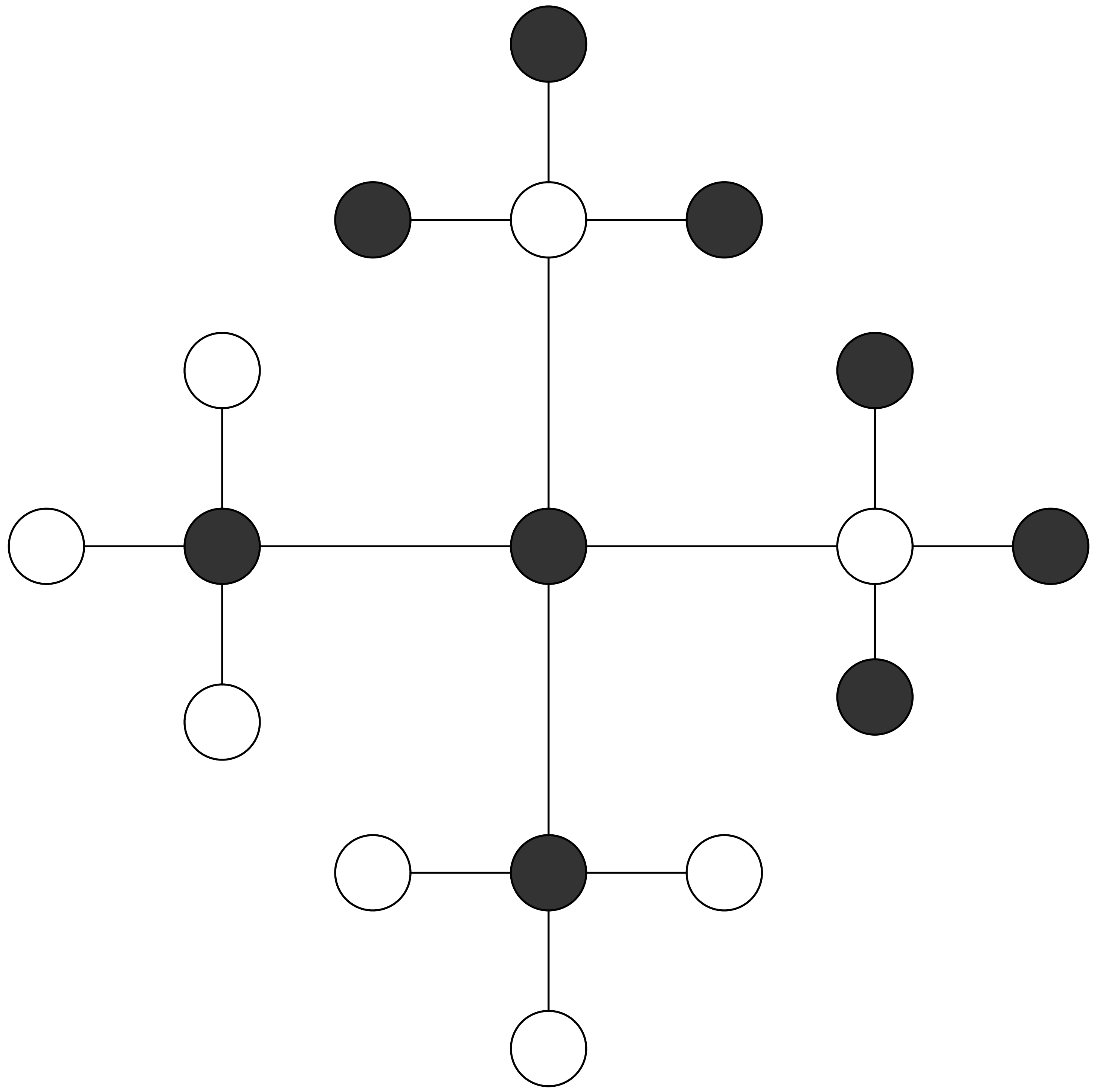
The distributed complexity of locally checkable problems on paths is decidable

Alkida Balliu, Sebastian Brandt,
Yi-Jun Chang, **Dennis Olivetti**, Mikaël Rabie, Jukka Suomela

General topic

Given a graph problem, can we **decide** its distributed time complexity?



LOCAL model

- Entities = **nodes**
- Communication links = **edges**
- Input graph = communication graph

LOCAL model

- Each node has a **unique identifier** from 1 to poly(n)
- **No bounds** on the computational power of the entities
- **No bounds** on the bandwidth

LOCAL model

- Round 0

LOCAL model

- Round 1

LOCAL model

- Round 2

LOCAL model

- After *t* **rounds**: knowledge of the graph up to **distance *t***
- Focus on **locality**

Locally Checkable Labelings (LCLs)

- **Input**
  - Graph of **constant** maximum degree Δ
  - Node labels from a **constant-size** set X

[Naor and Stockmeyer, 1995]

Locally Checkable Labelings (LCLs)

- **Input**
  - Graph of **constant** maximum degree Δ
  - Node labels from a **constant-size** set X
- **Output**
  - Node labels from a **constant-size** set Y, such that each node satisfies some **local constraints**

[Naor and Stockmeyer, 1995]

Locally Checkable Labelings (LCLs)

- **Input**
  - Graph of **constant** maximum degree Δ
  - Node labels from a **constant-size** set X
- **Output**
  - Node labels from a **constant-size** set Y, such that each node satisfies some **local constraints**
- **Correctness**
  - A solution is globally correct if it is correct in all **constant-radius** neighborhoods

[Naor and Stockmeyer, 1995]

Example: weak 2-coloring

- **Output**: color nodes from a palette of **2 colors**
- **Constraint**: each node must have a **different color** from **at least 1** neighbor

Objective of this work

Given an LCL **Π = (input, output, constraints)** we want to:

- **Decide** the distributed complexity of **Π**
- **Synthesize** an asymptotically optimal algorithm for **Π**

State of the art

State of the art

- **Paths/Cycles with NO input:**
  - the time complexity is always decidable, and
  - it can be either $O(1)$, $\Theta(\log^* n)$, or $\Theta(n)$

    [Naor and Stockmeyer 1995] [Chang et al. 2016] [Brandt et al. 2017]

State of the art

- **Paths/Cycles with NO input:**
  - the time complexity is always decidable, and
  - it can be either **$O(1)$, $\Theta(\log^* n)$,** or **$\Theta(n)$**

    [Naor and Stockmeyer 1995] [Chang et al. 2016] [Brandt et al. 2017]
- **$\sqrt{n} \times \sqrt{n}$ Grids:**
  - the time complexity is undecidable, but
  - if the grid has no input, it can be either **$O(1)$, $\Theta(\log^* n)$,** or **$\Theta(\sqrt{n})$**
  - if the grid has no input and is toroidal, it is decidable if there is a **$O(1)$** algorithm

    [Naor and Stockmeyer 1995] [Brandt et al. 2017]

State of the art

- **Paths/Cycles with NO input:**
  - the time complexity is always decidable, and
  - it can be either **$O(1)$, $\Theta(\log^* n)$,** or **$\Theta(n)$**

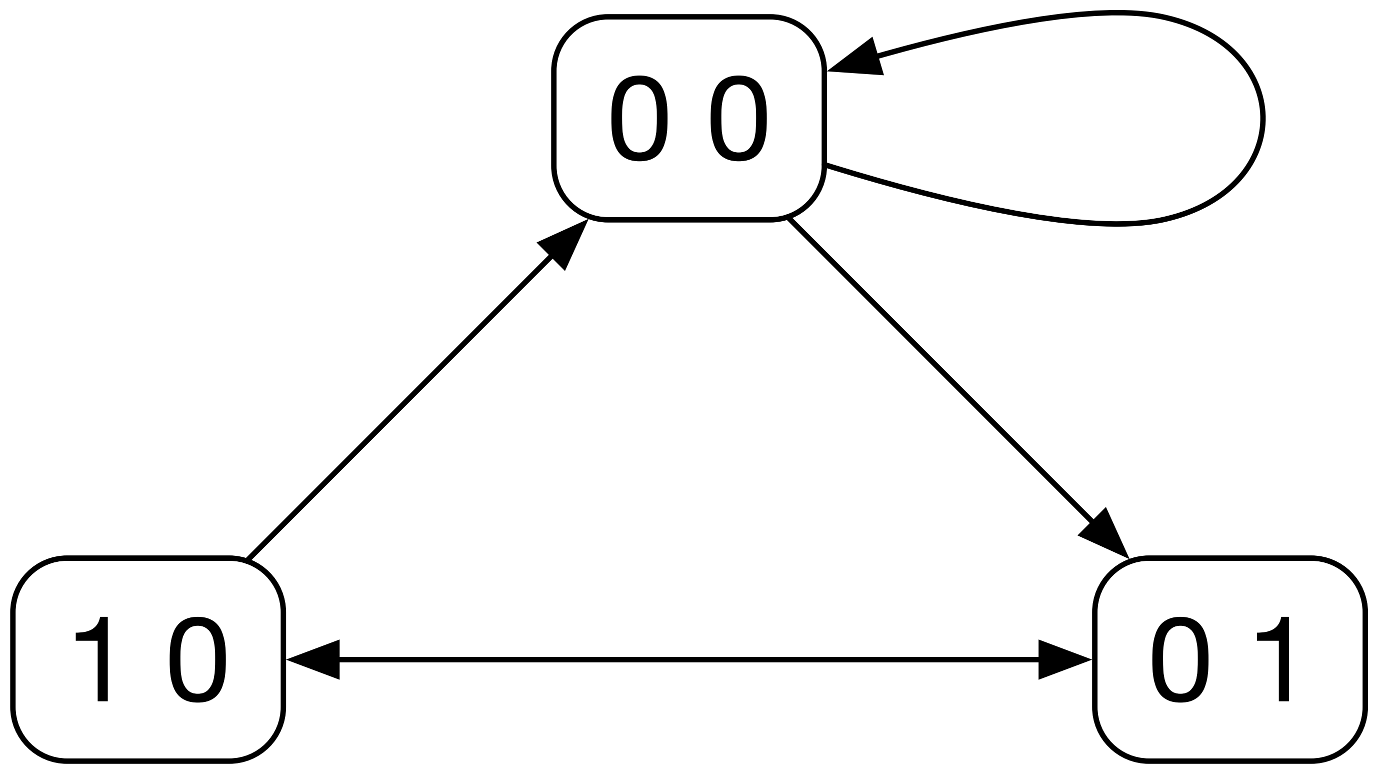
    [Naor and Stockmeyer 1995] [Chang et al. 2016] [Brandt et al. 2017]
- **$\sqrt{n} \times \sqrt{n}$ Grids:**
  - the time complexity is undecidable, but
  - if the grid has no input, it can be either **$O(1)$, $\Theta(\log^* n)$,** or **$\Theta(\sqrt{n})$**
  - if the grid has no input and is toroidal, it is decidable if there is a **$O(1)$** algorithm

    [Naor and Stockmeyer 1995] [Brandt et al. 2017]
- **Trees:**
  - it is decidable if the LCL requires **$O(\log n)$** or **$n^{\Omega(1)}$**

    [Chang and Pettie 2017]

Unlabeled Directed Cycles

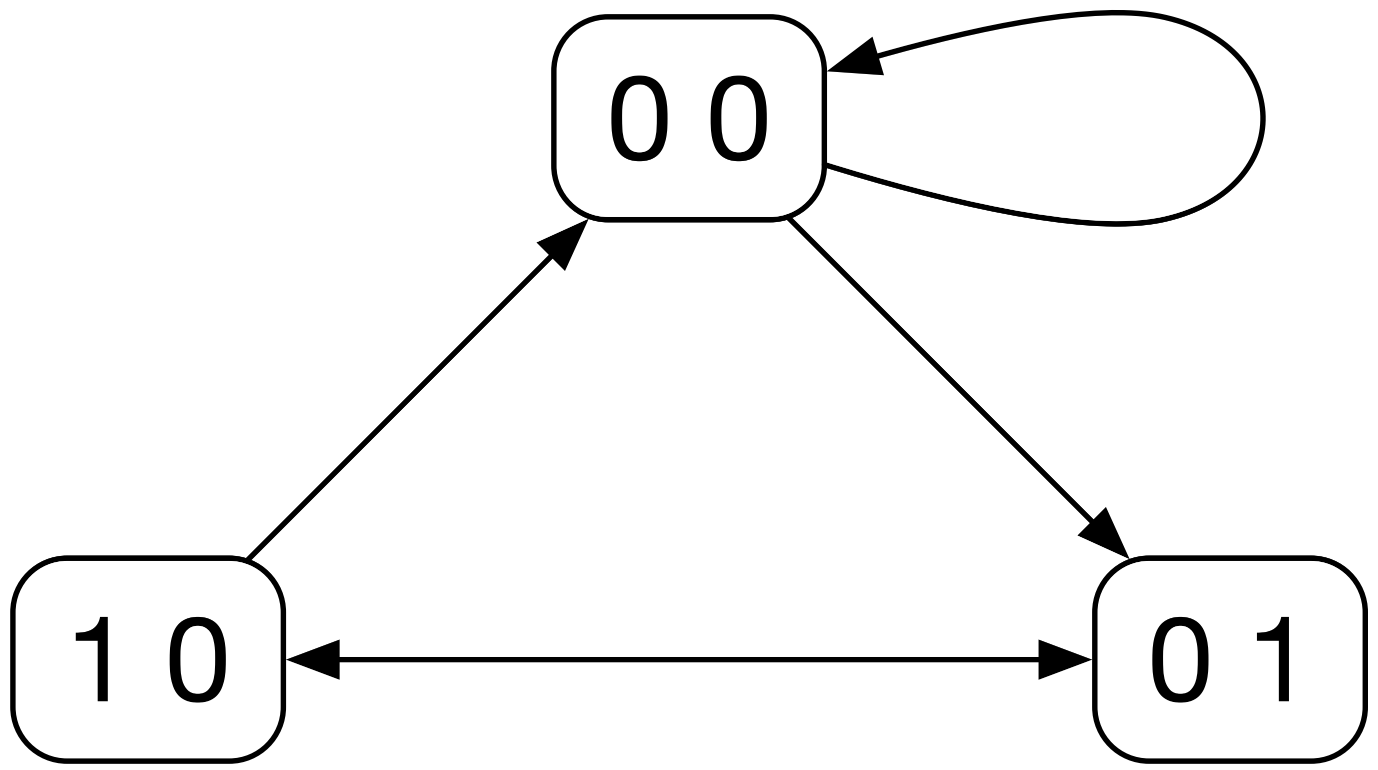
Independent Set

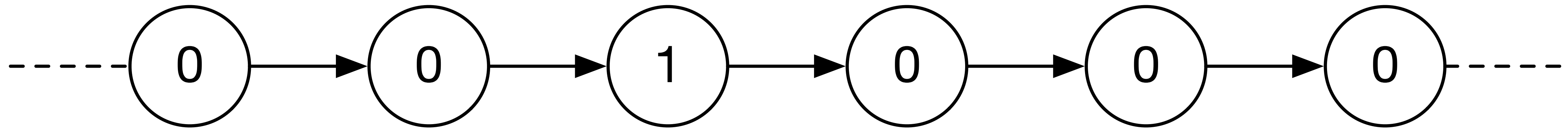


[Brandt et al. 2017]

Unlabeled Directed Cycles

Independent Set

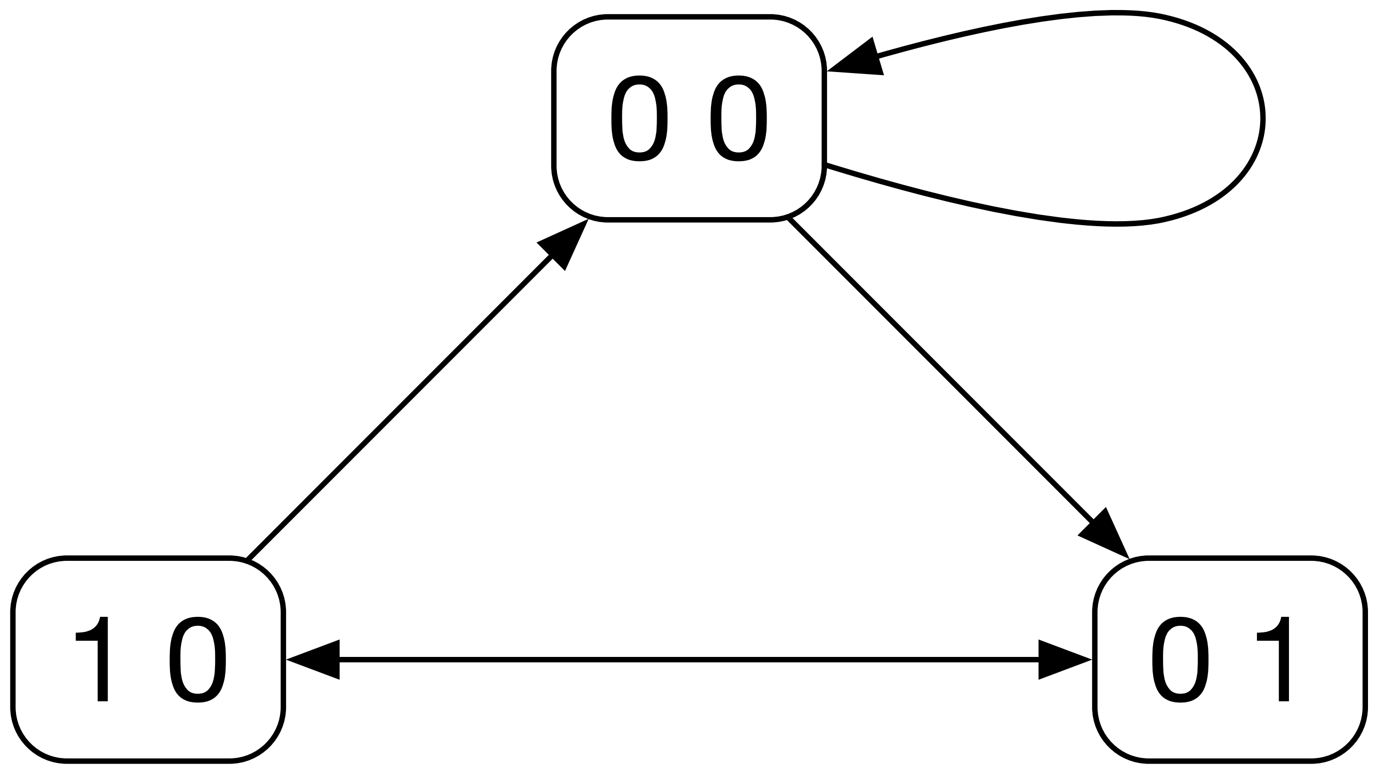


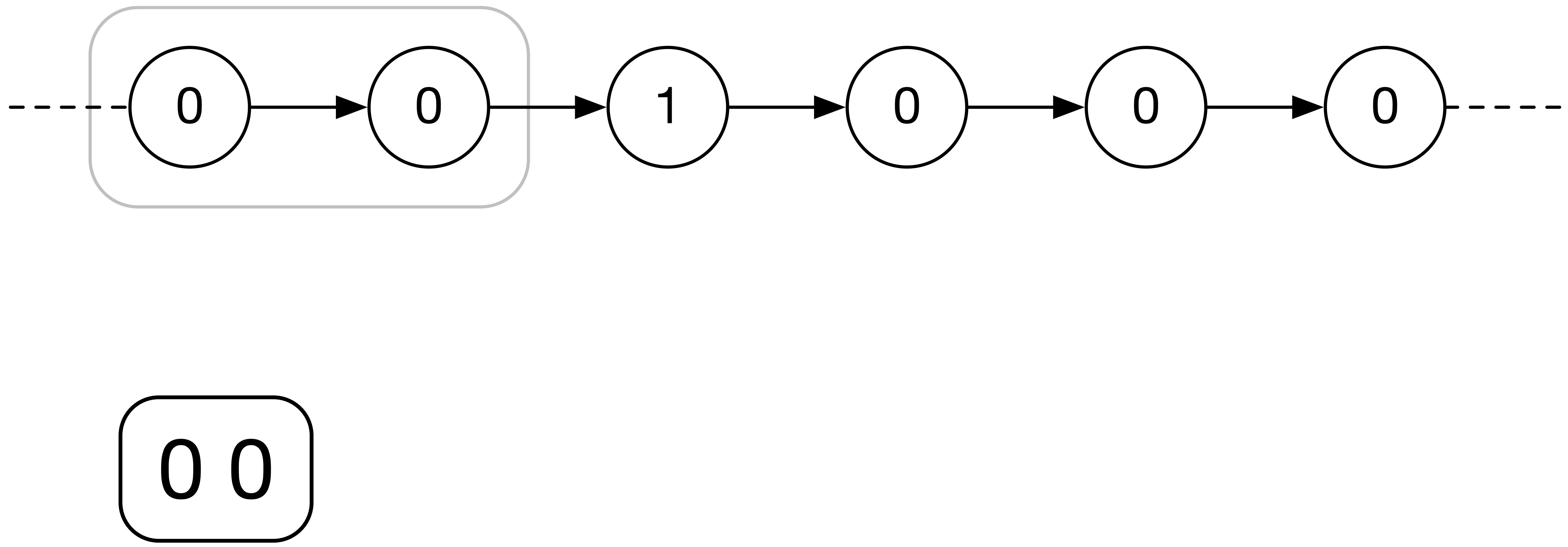


[Brandt et al. 2017]

Unlabeled Directed Cycles

Independent Set

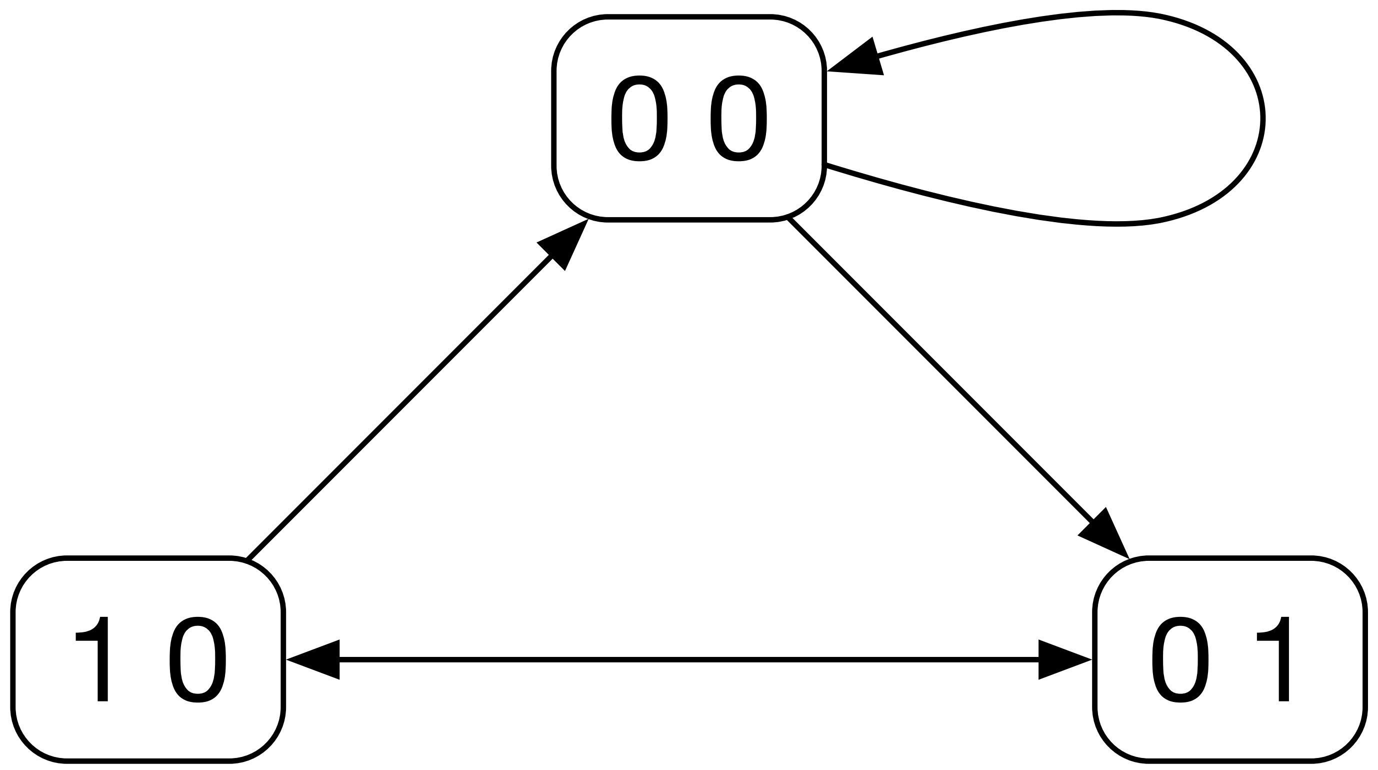


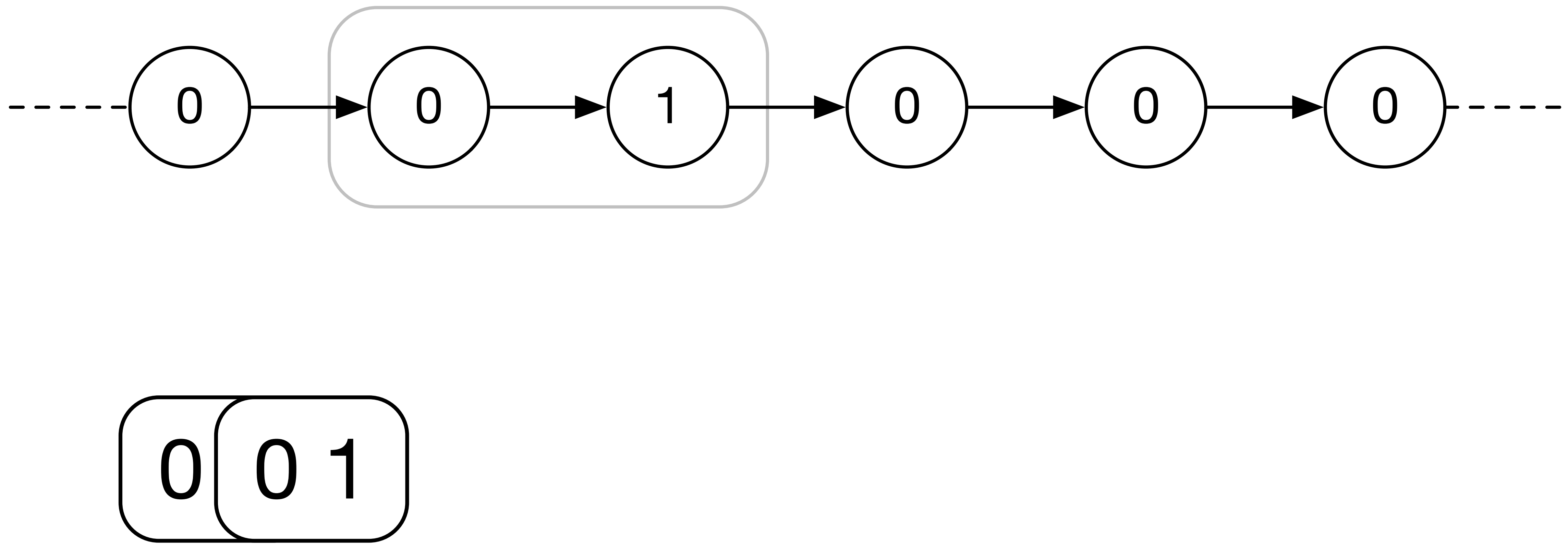


[Brandt et al. 2017]

Unlabeled Directed Cycles

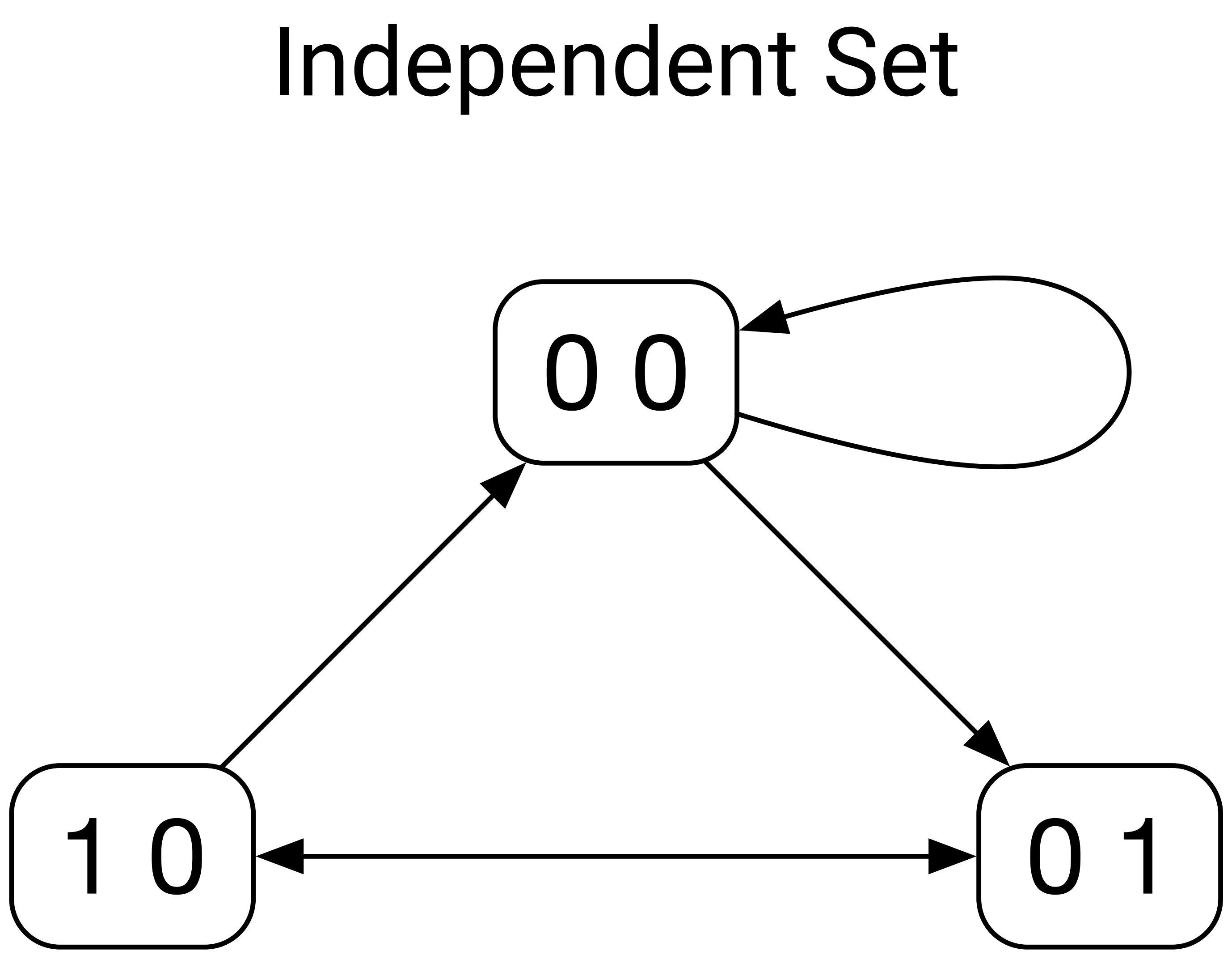
Independent Set

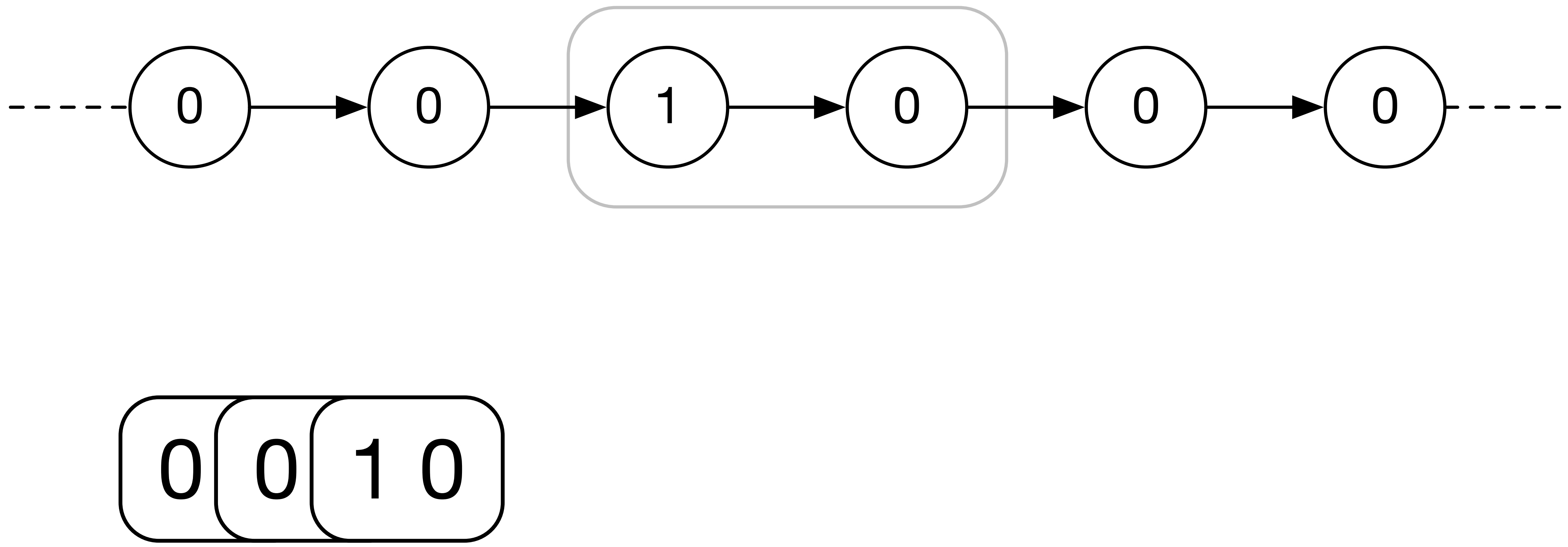




[Brandt et al. 2017]

Unlabeled Directed Cycles

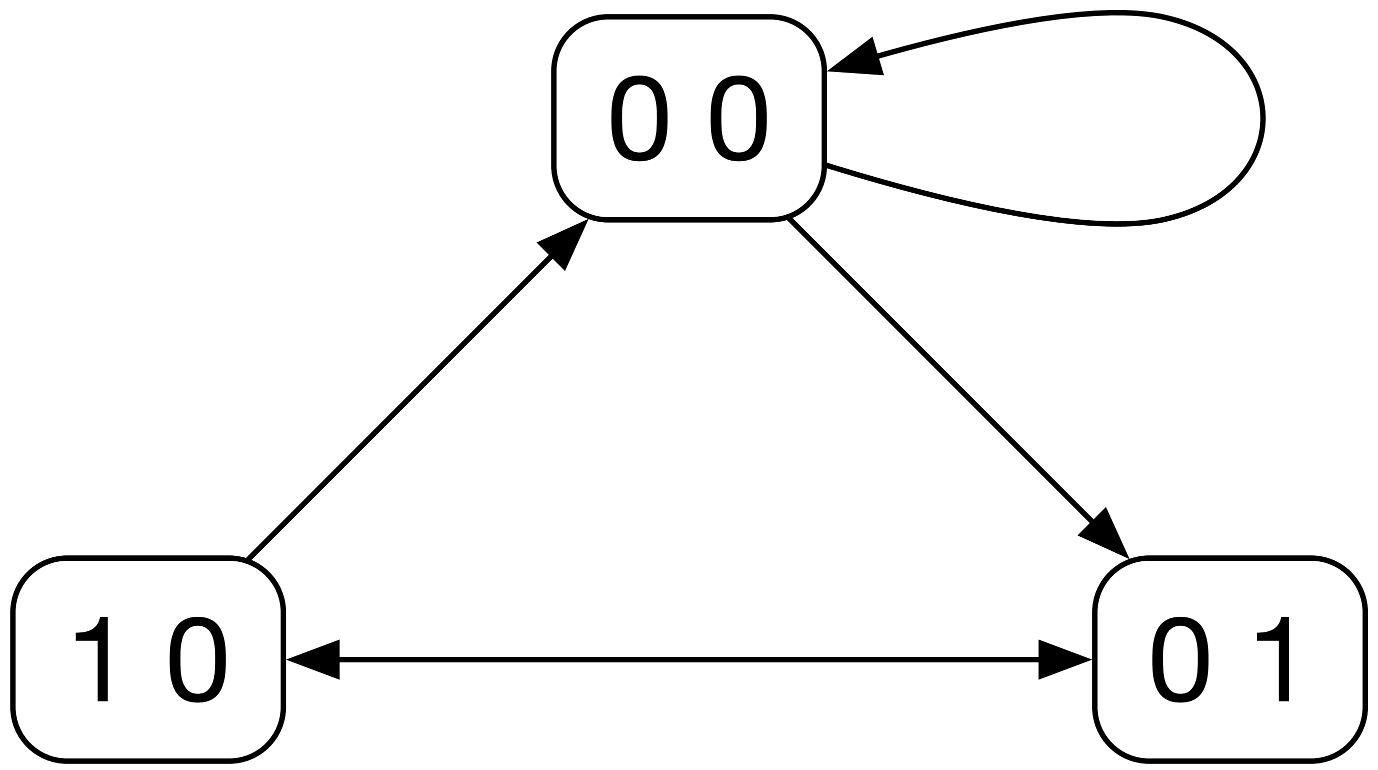


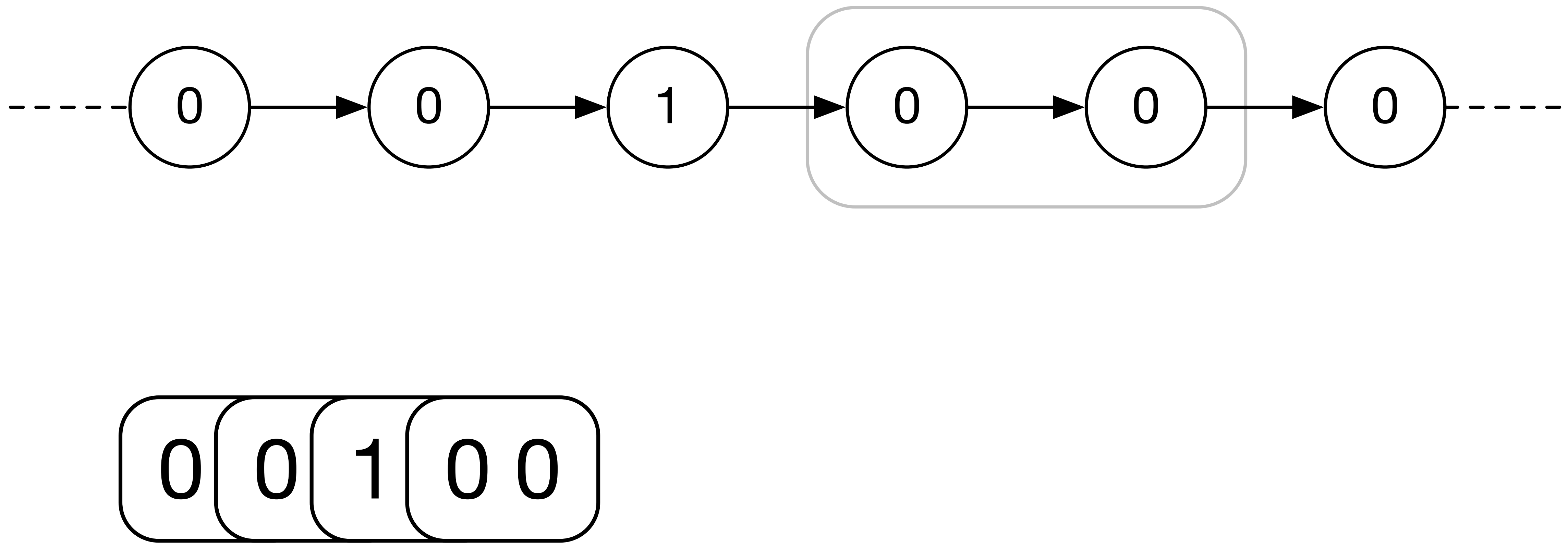


[Brandt et al. 2017]

Unlabeled Directed Cycles

Independent Set

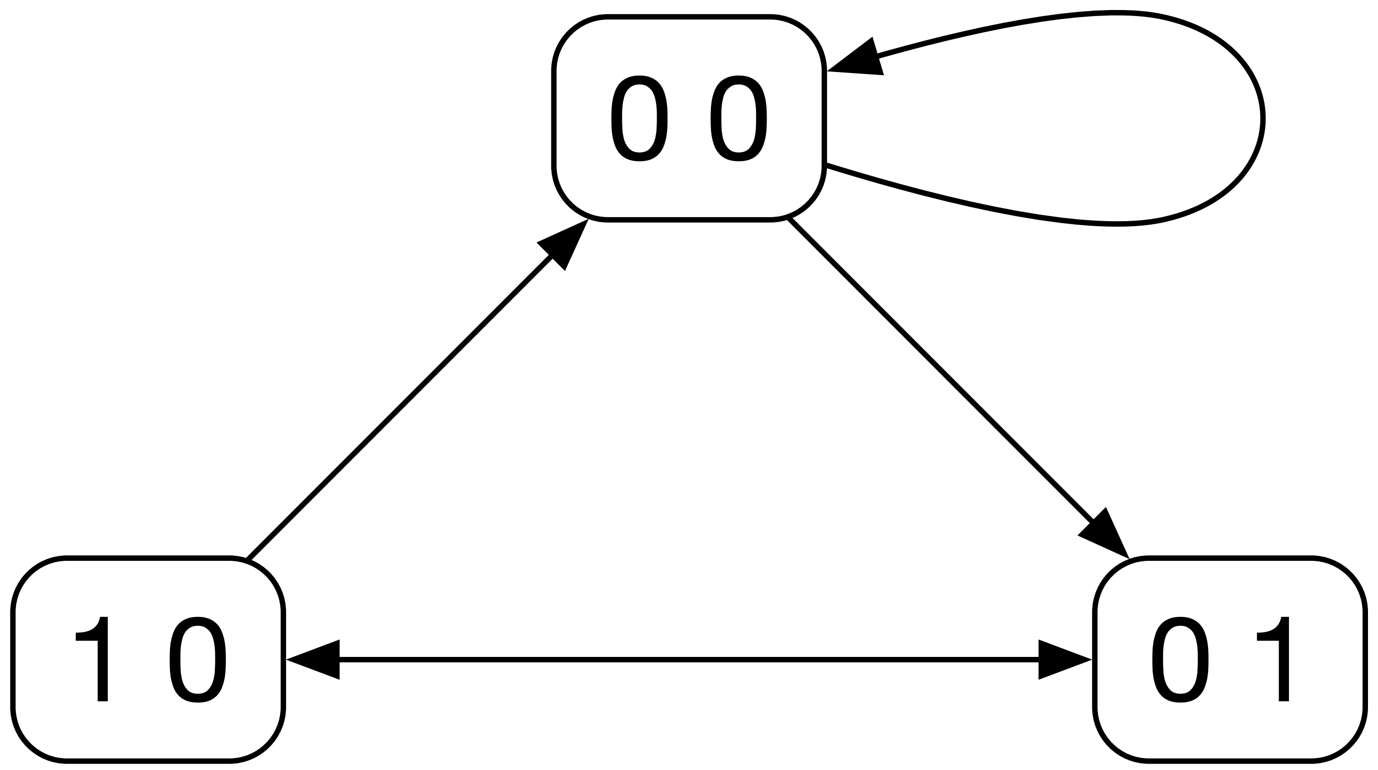


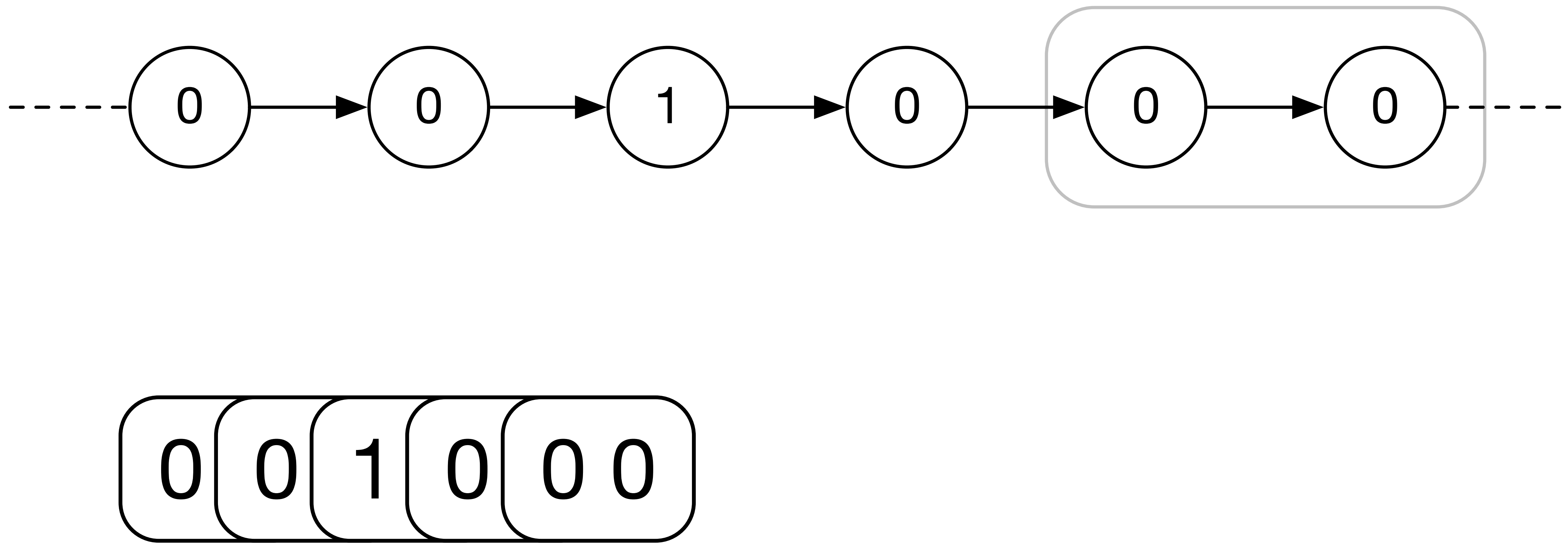


[Brandt et al. 2017]

Unlabeled Directed Cycles

Independent Set

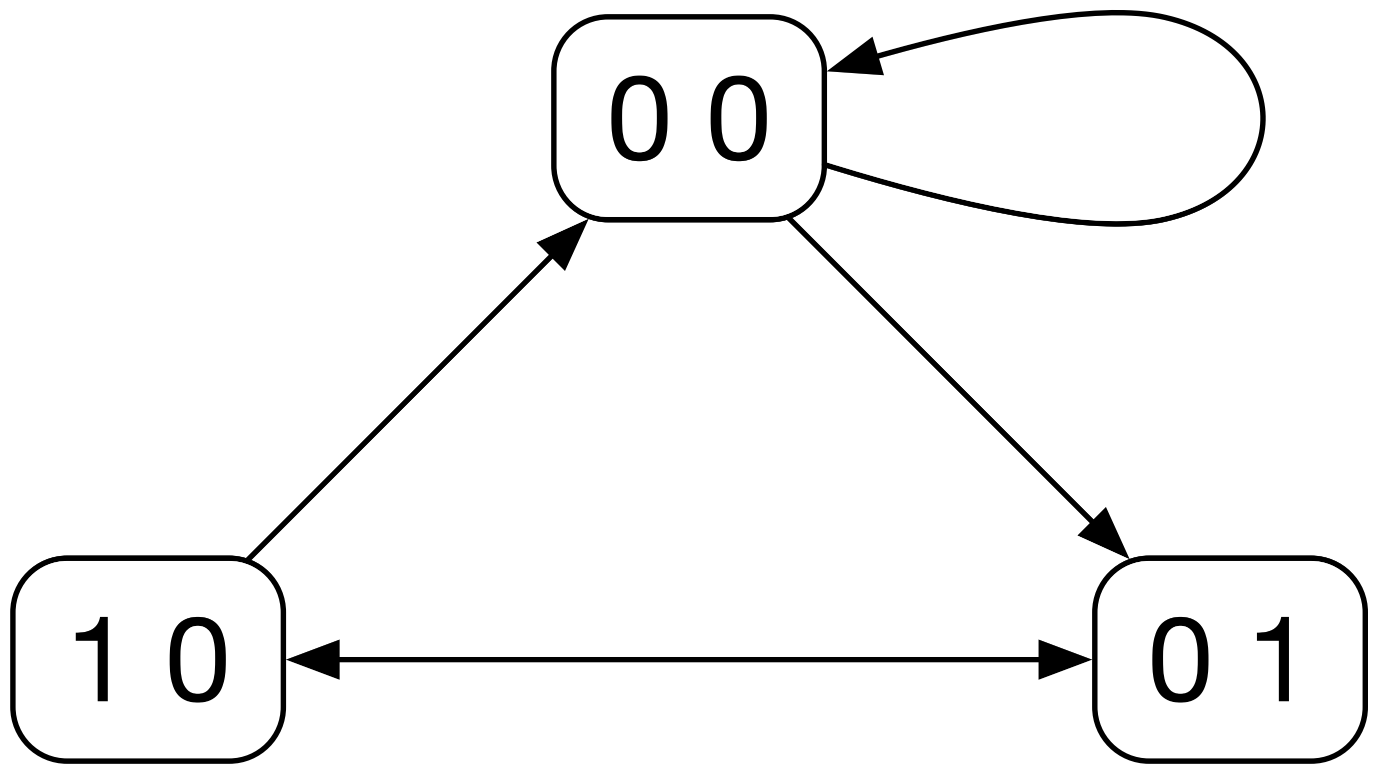




[Brandt et al. 2017]

Unlabeled Directed Cycles

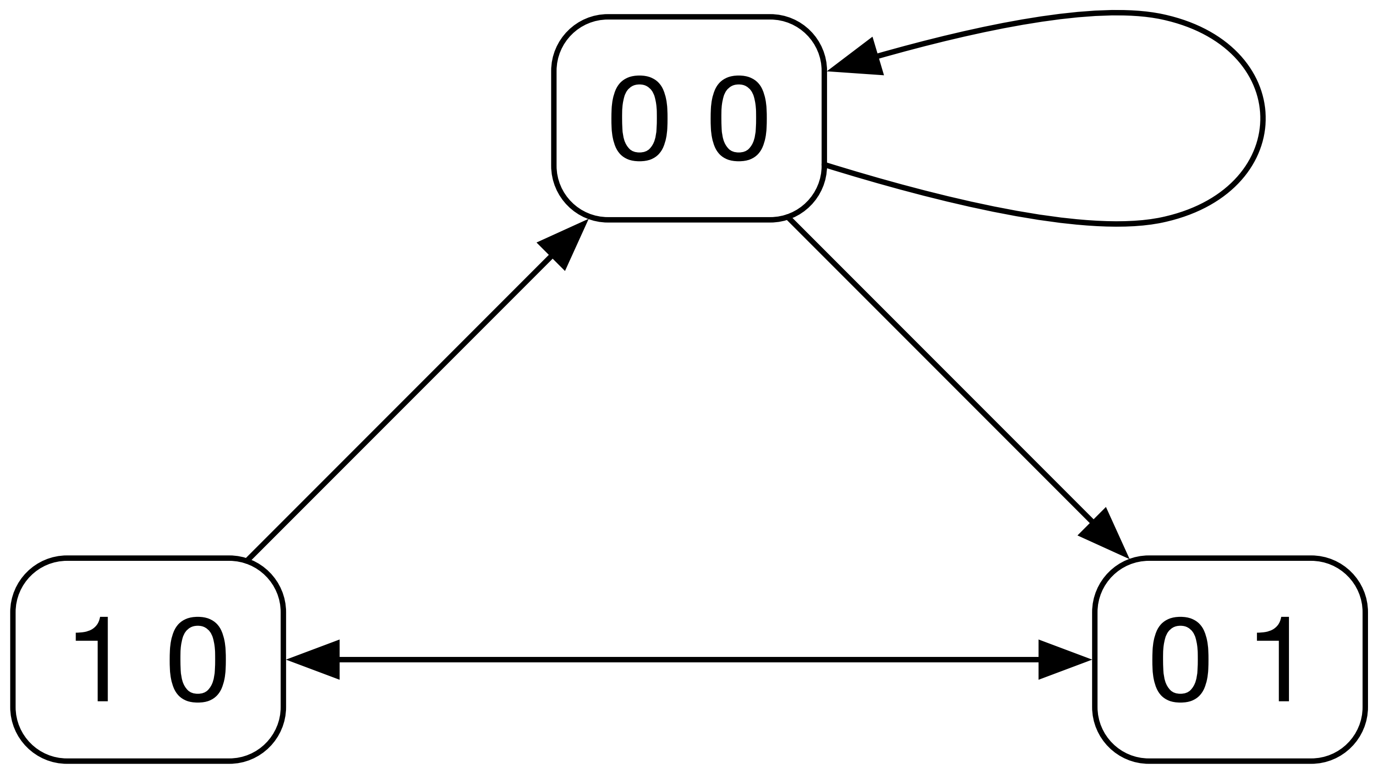
Independent Set



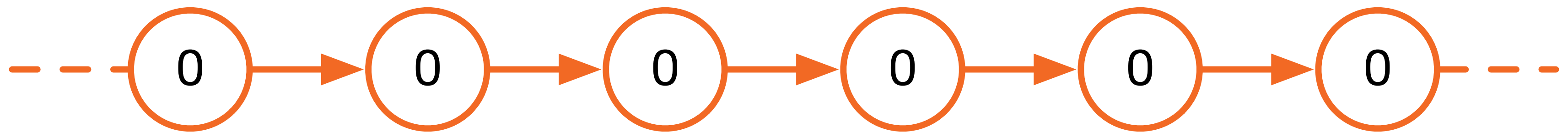
[Brandt et al. 2017]

Unlabeled Directed Cycles

Independent Set



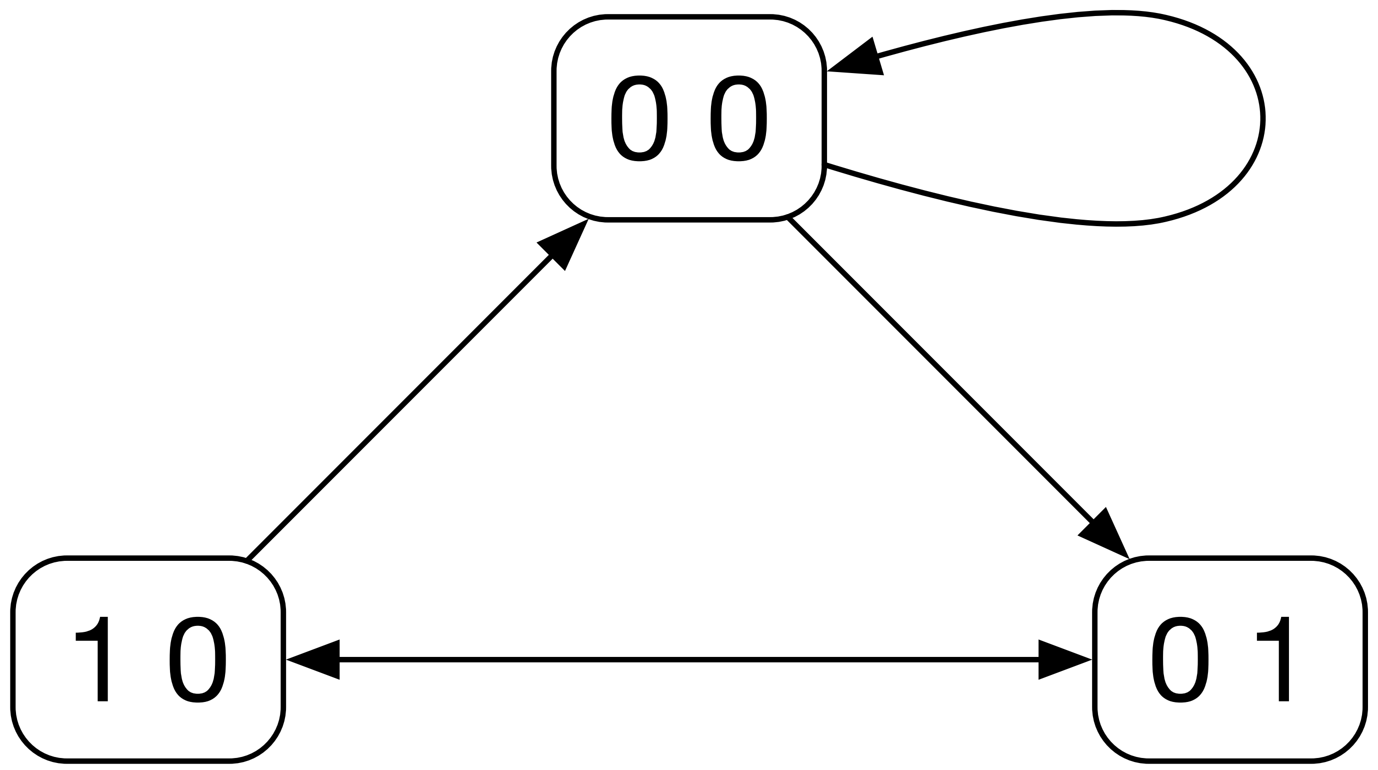
Self loop: **O(*1*)**



[Brandt et al. 2017]

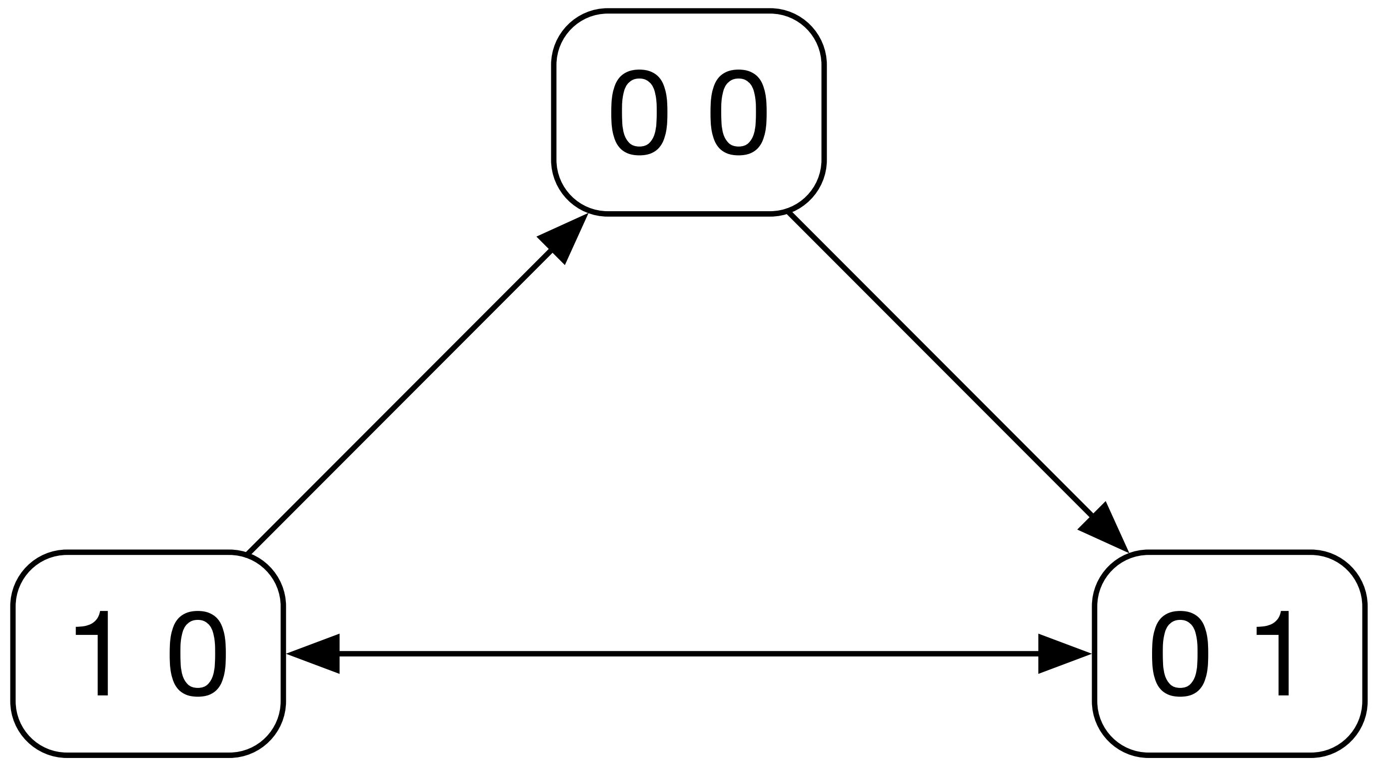
Unlabeled Directed Cycles

Independent Set



Self loop: **O(*1*)**

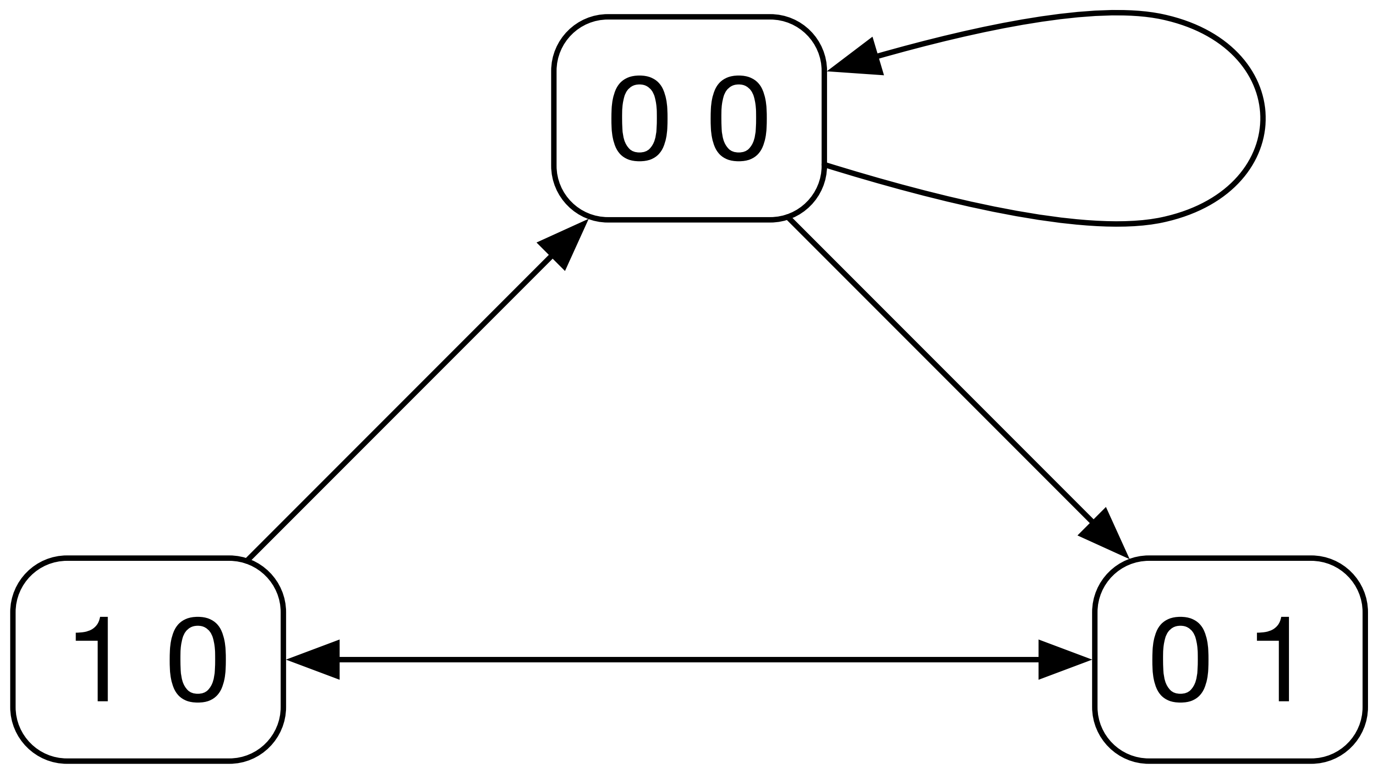
Maximal Independent Set



[Brandt et al. 2017]

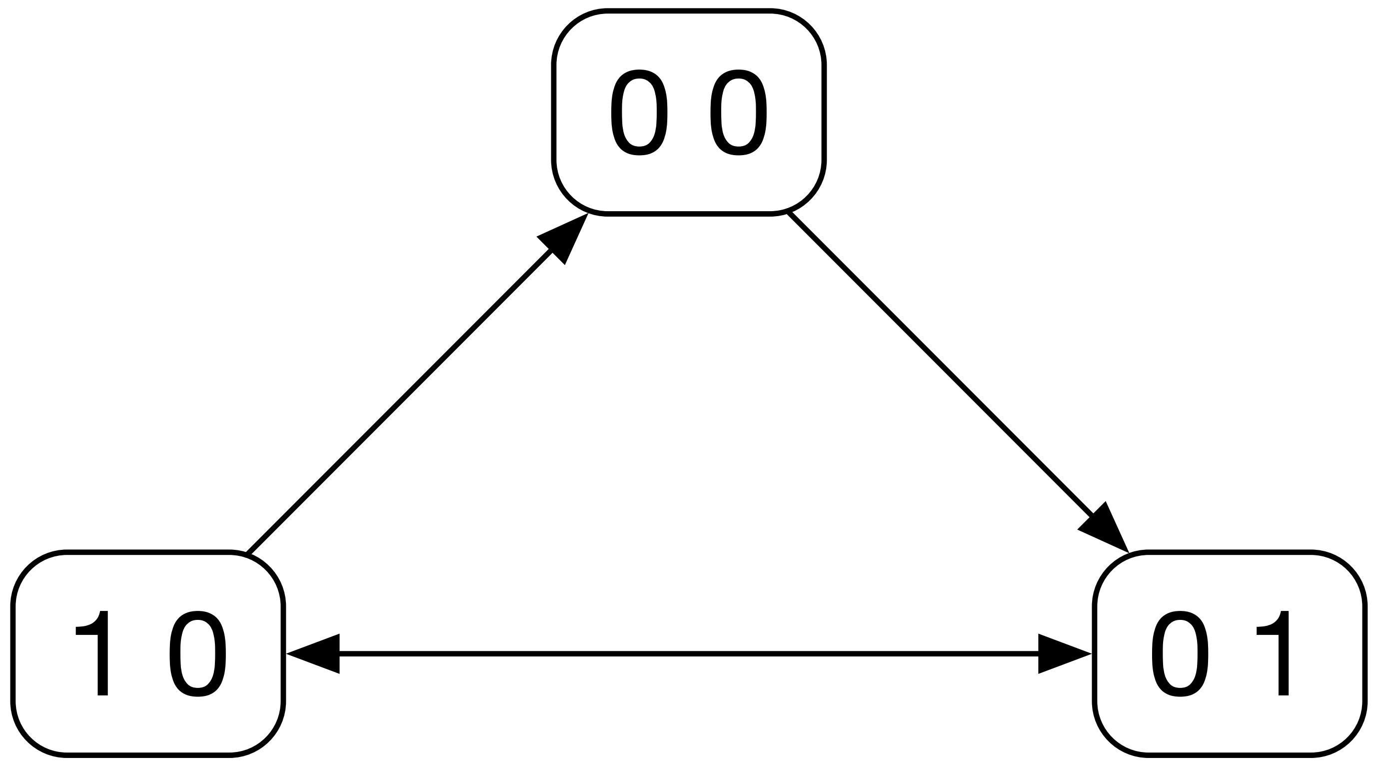
Unlabeled Directed Cycles

Independent Set



Self loop: **$O(1)$**

Maximal Independent Set



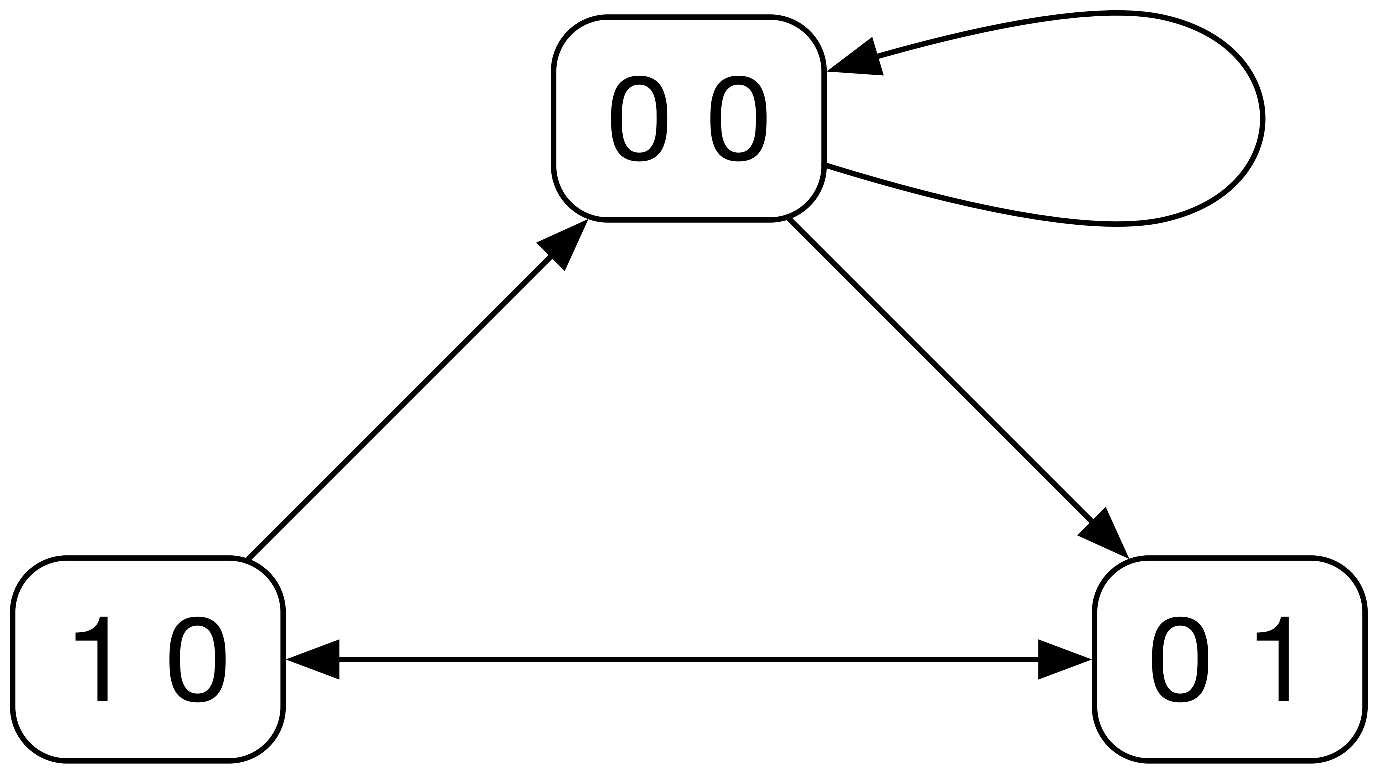
Flexible state: **$\Theta(\log^* n)$**

"1 0" is flexible:
∀ k ≥ 3, ∃ cycle of length k
that starts and ends at "1 0"

[Brandt et al. 2017]

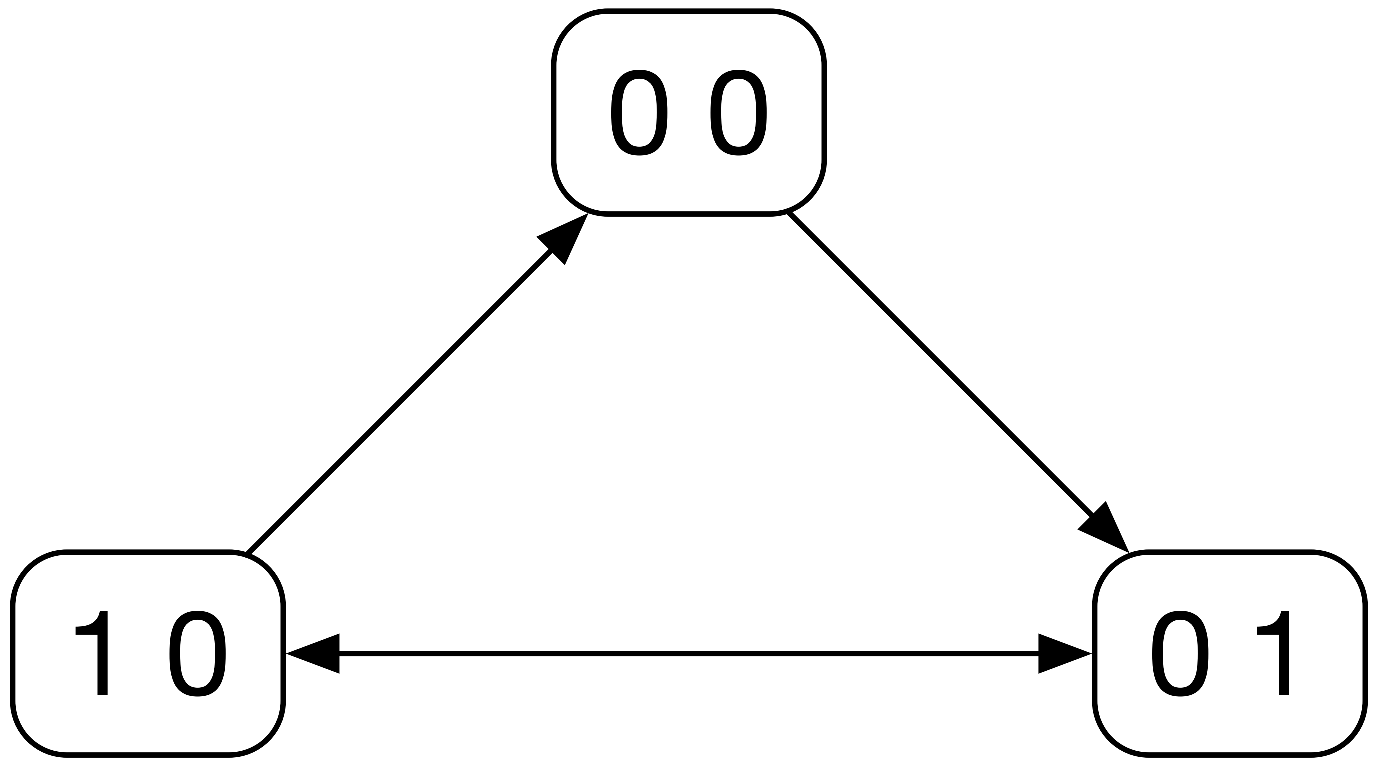
Unlabeled Directed Cycles

Independent Set



Self loop: **O(*1*)**

Maximal Independent Set



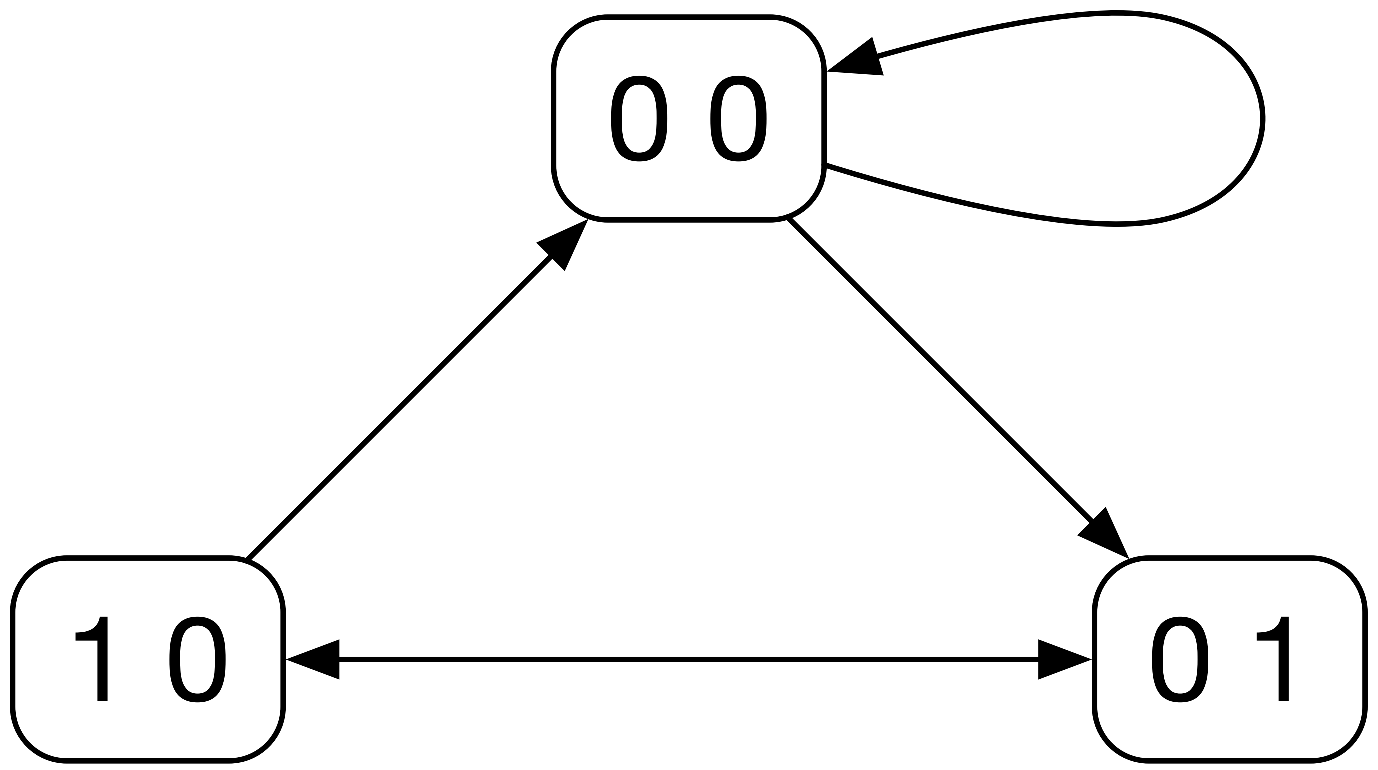
Flexible state: **Θ(log* *n*)**

"1 0" is flexible:
∀ k ≥ 3, ∃ cycle of length k
that starts and ends at "1 0"

[Brandt et al. 2017]

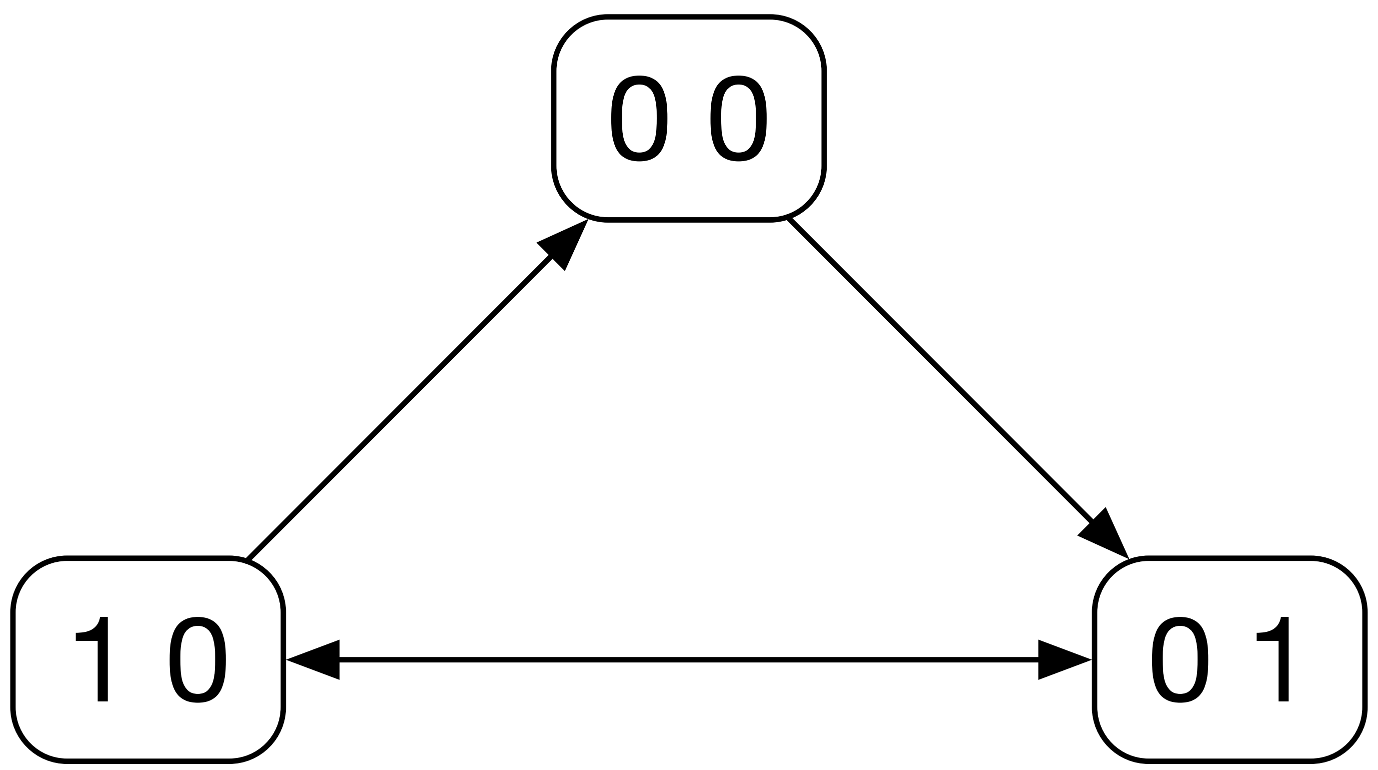
Unlabeled Directed Cycles

Independent Set



Self loop: **O(*1*)**

Maximal Independent Set



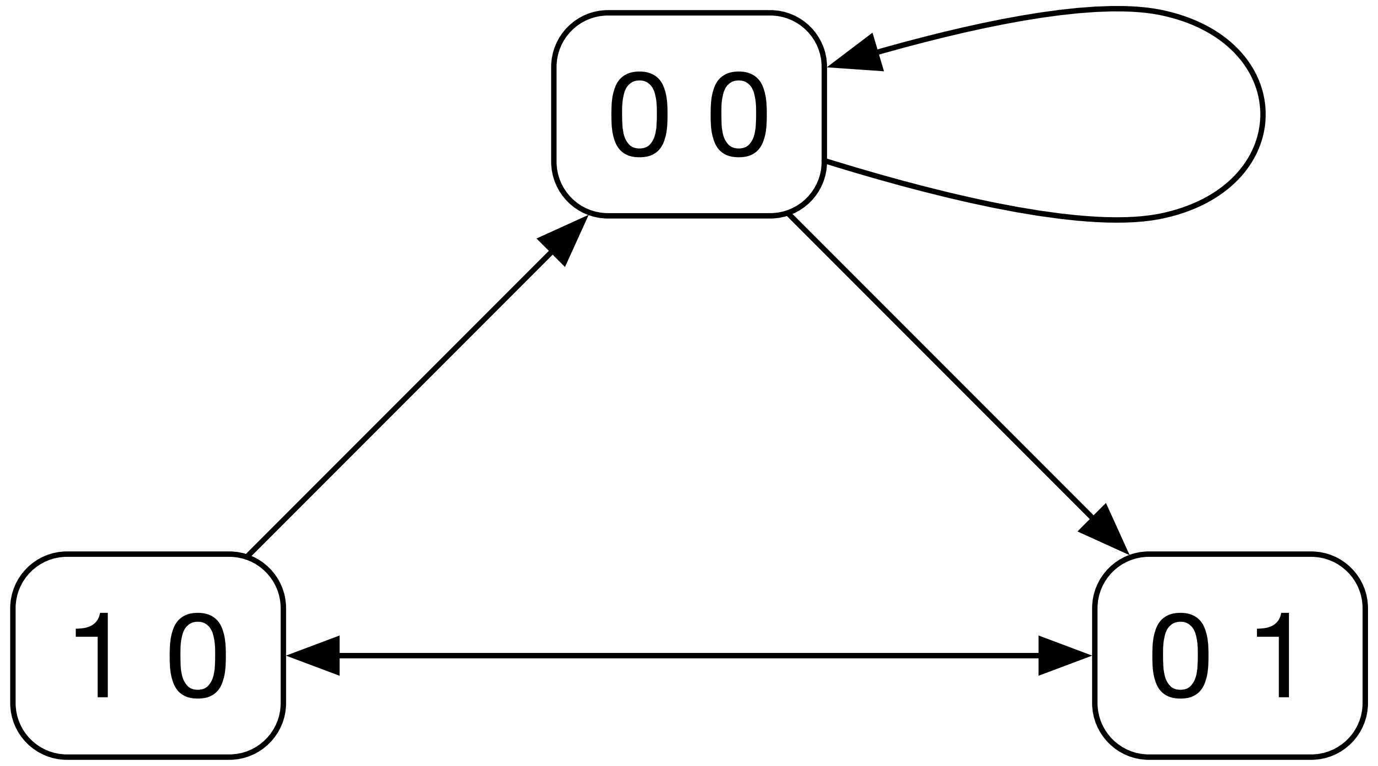
Flexible state: **Θ(log* *n*)**

"1 0" is flexible:
∀ k ≥ 3, ∃ cycle of length k
that starts and ends at "1 0"

[Brandt et al. 2017]

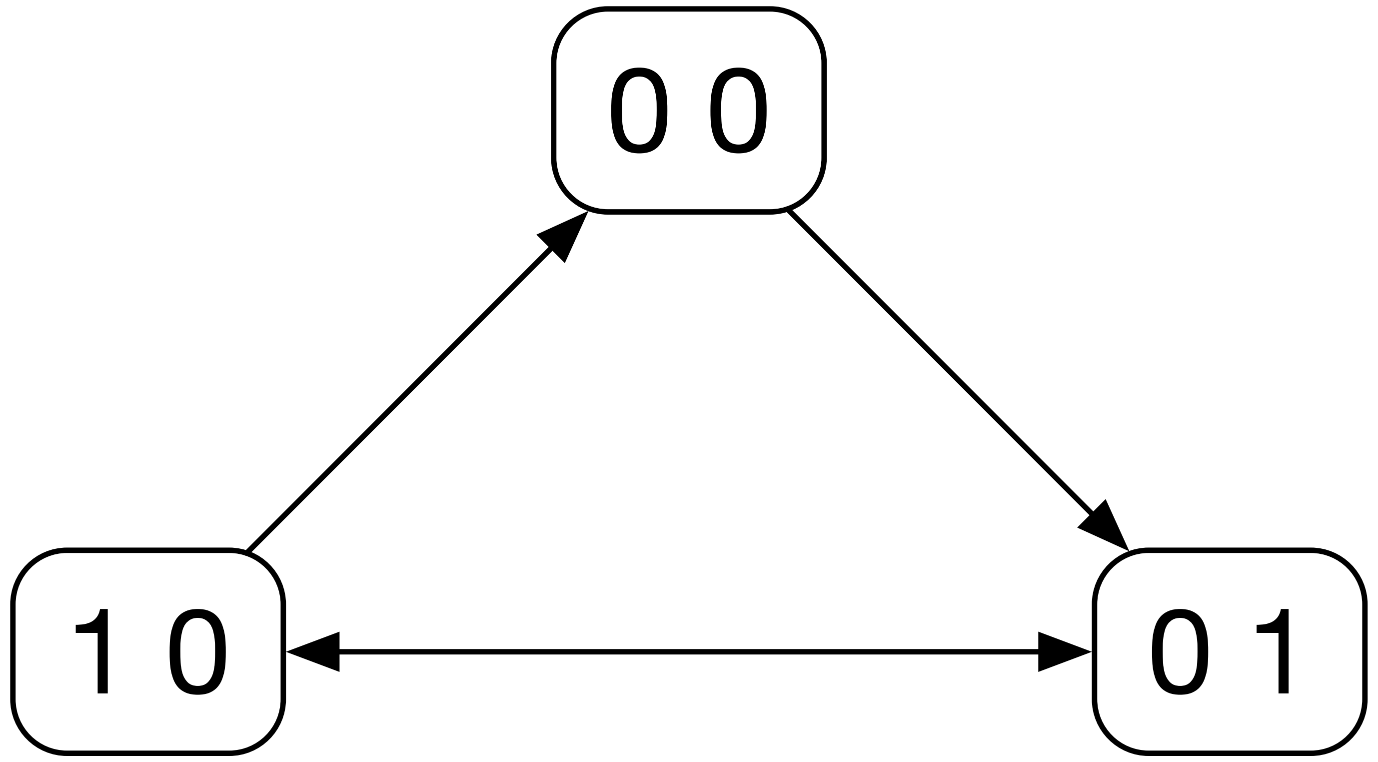
Unlabeled Directed Cycles

Independent Set



Self loop: **O(*1*)**

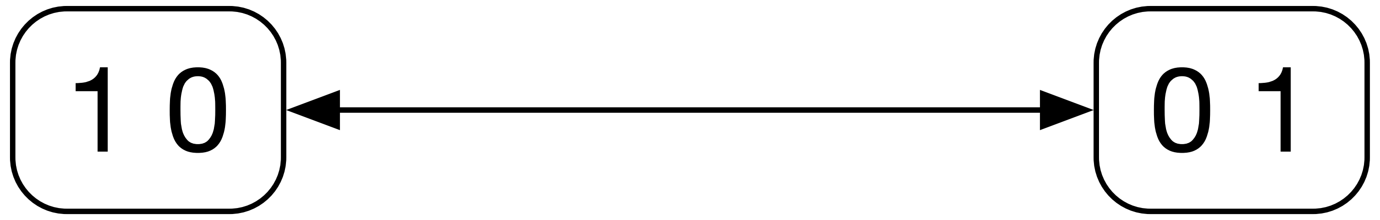
Maximal Independent Set



Flexible state: **Θ(log* *n*)**

"1 0" is flexible:
∀ k ≥ 3, ∃ cycle of length k
that starts and ends at "1 0"

2-Coloring



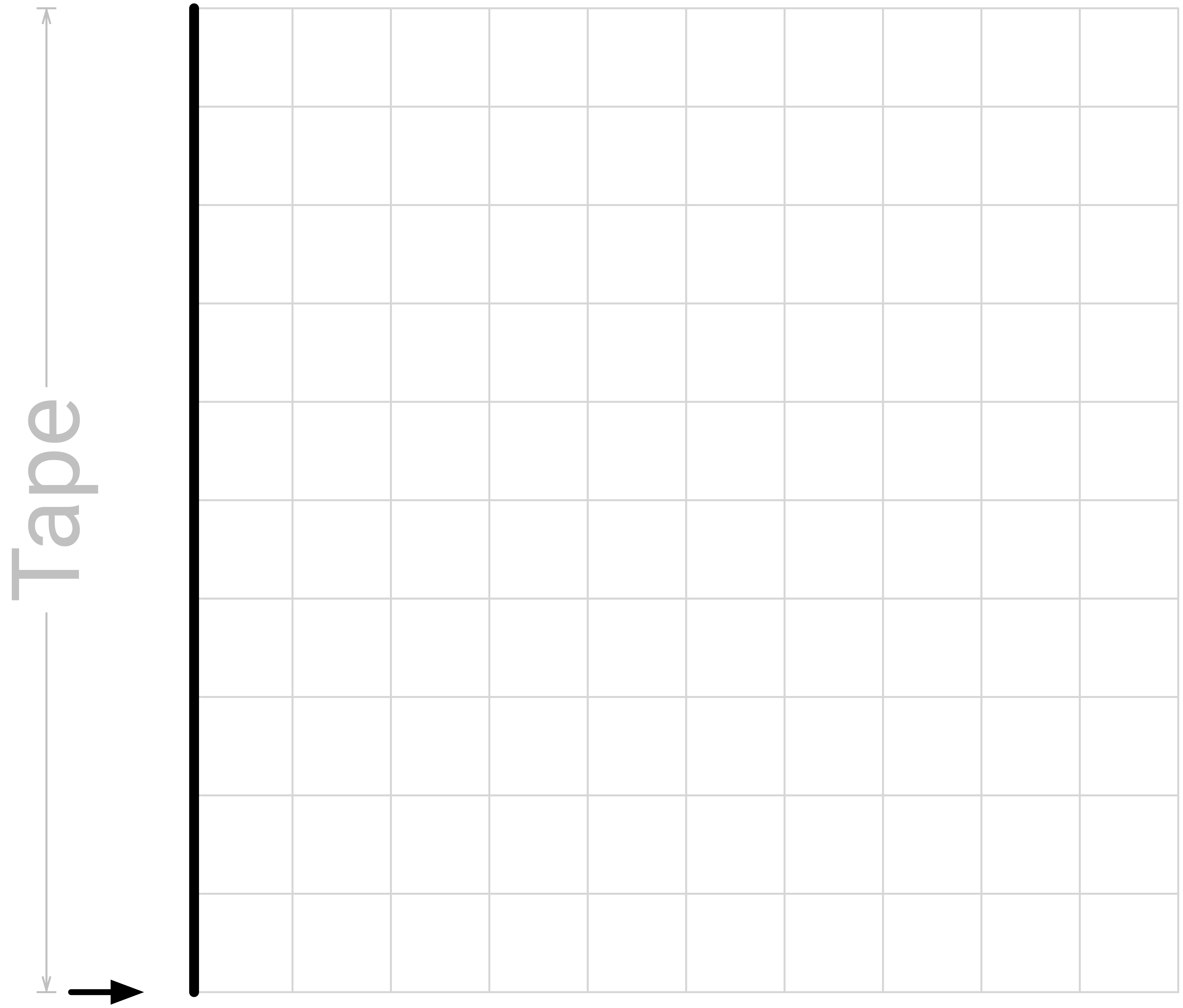
Otherwise: **Ω(*n*)**

[Brandt et al. 2017]

Grids

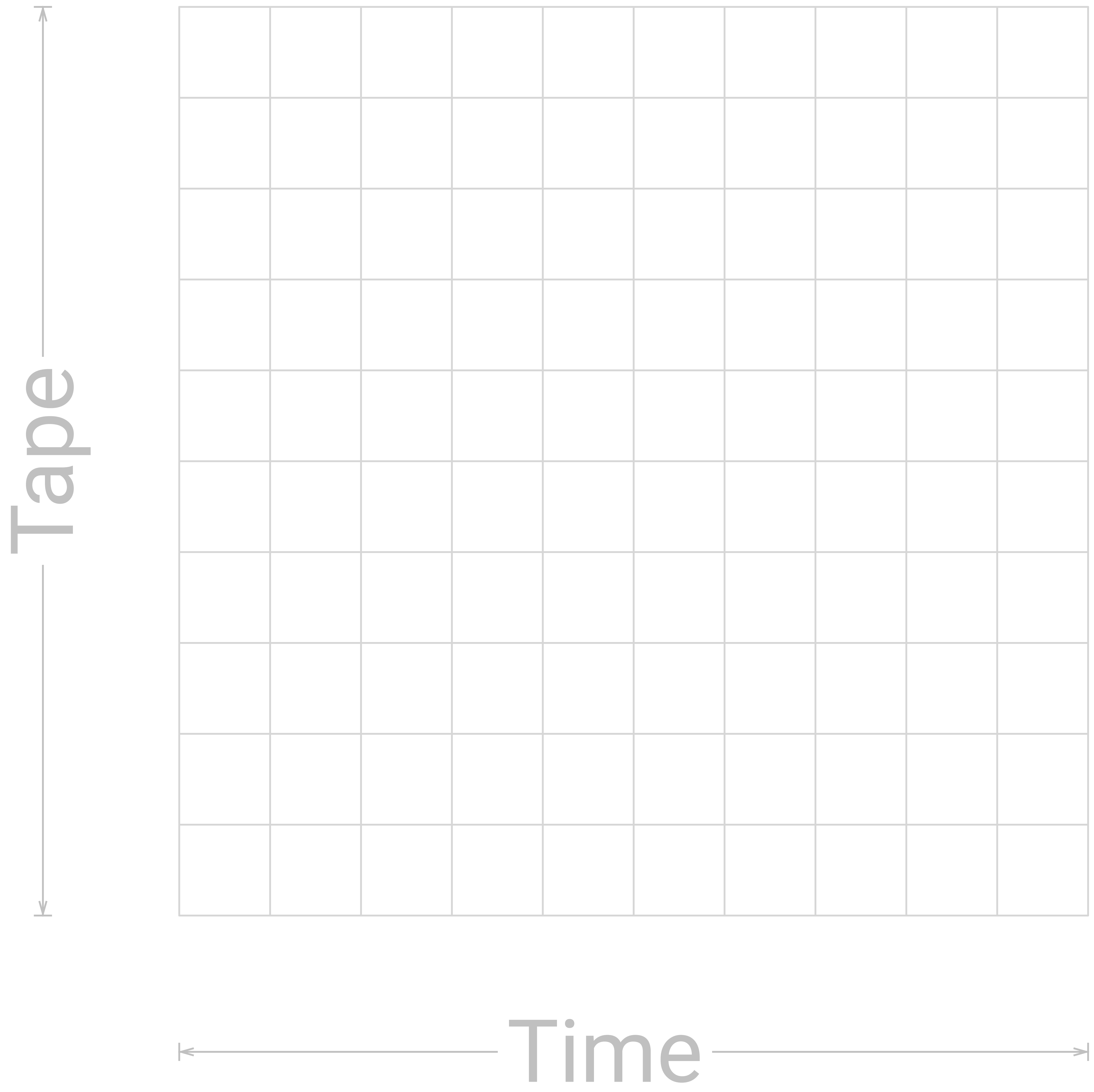
[Naor and Stockmeyer 1995]

Grids



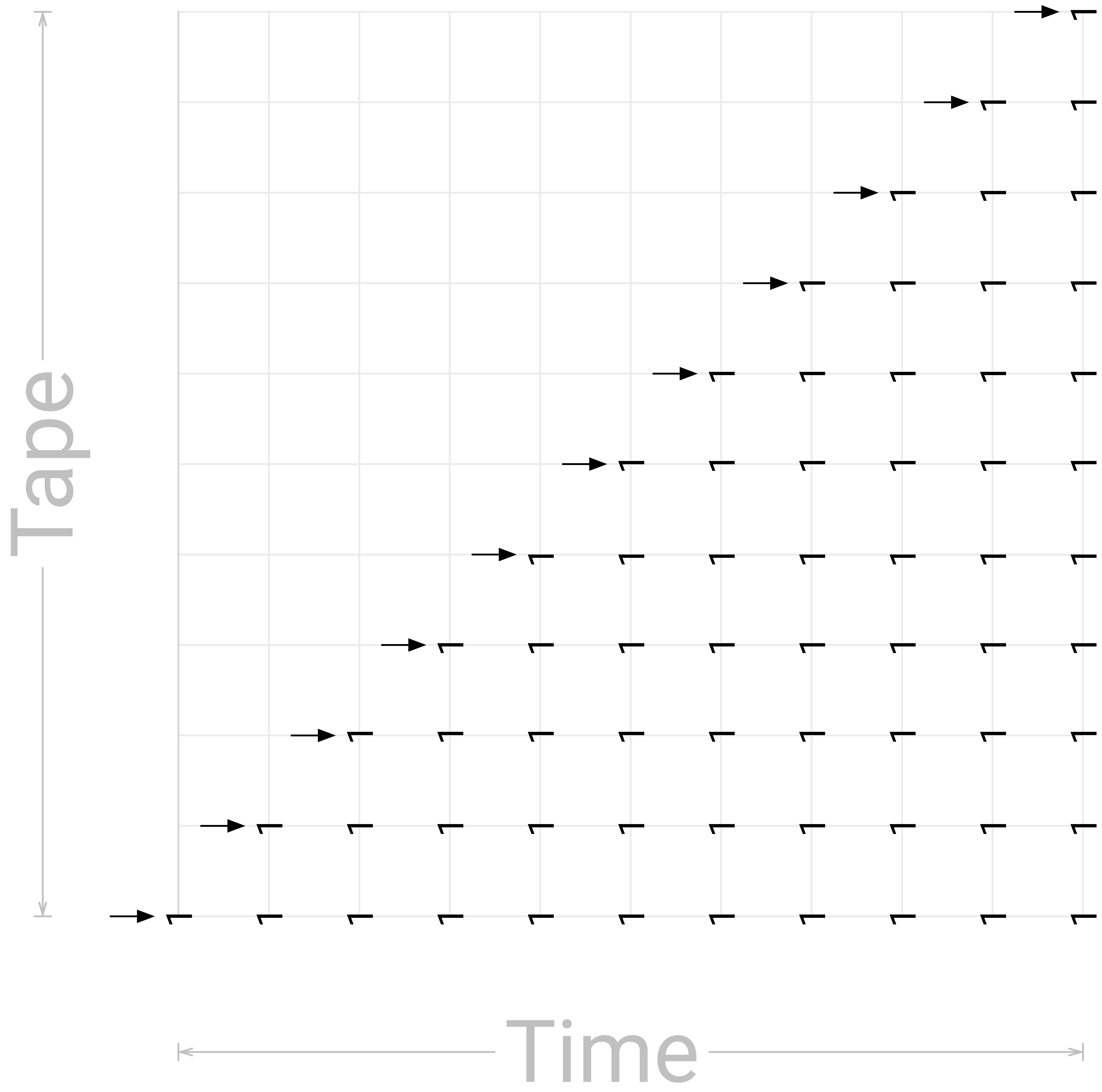
[Naor and Stockmeyer 1995]

Grids



[Naor and Stockmeyer 1995]

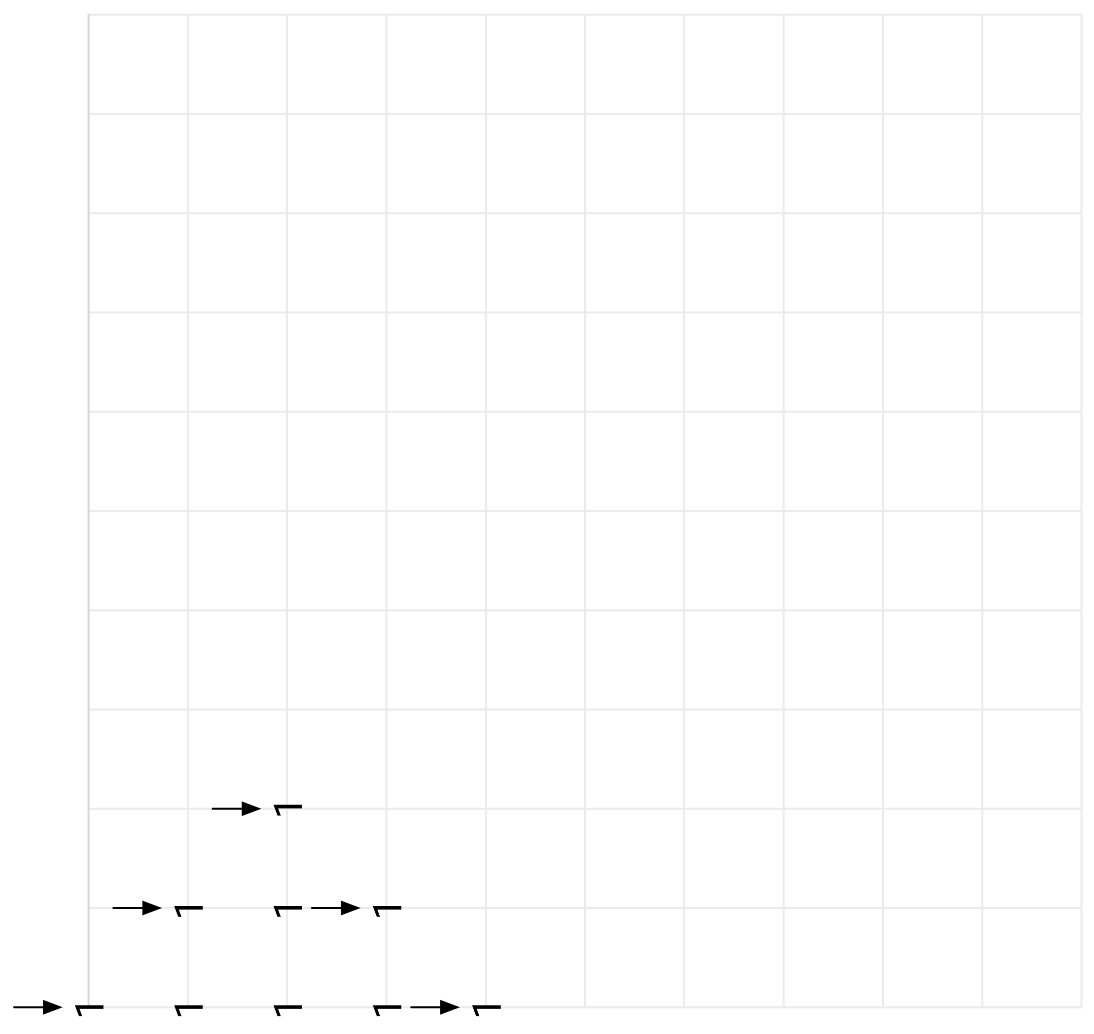
Grids

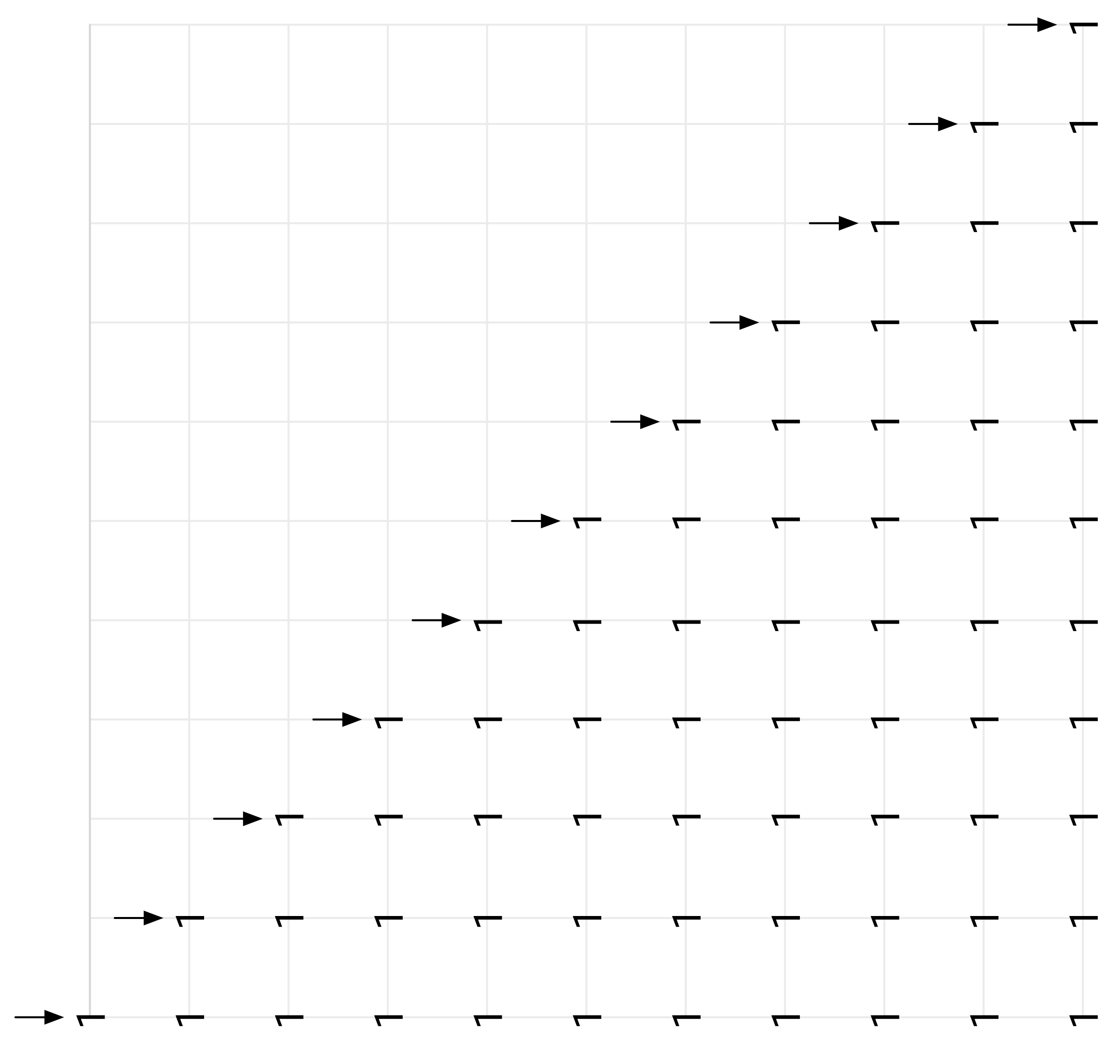


[Naor and Stockmeyer 1995]

Grids

- Define an LCL that requires to output the execution of a Turing machine
- If the machine ***terminates***, the LCL can be solved in $O(1)$
- If the machine ***does not terminate***, the LCL requires $\Omega(\sqrt{n})$





[Naor and Stockmeyer 1995]

General Picture

General Picture

- Grids allow to propagate too much information

General Picture

- Grids allow to propagate too much information
- On trees/bounded treewidth graphs it should not be possible

General Picture

- Grids allow to propagate too much information
- On trees/bounded treewidth graphs it should not be possible
- Let us prove that the complexity of LCLs is decidable on trees!

General Picture

- Grids allow to propagate too much information
- On trees/bounded treewidth graphs it should not be possible
- Let us prove that the complexity of LCLs is decidable on trees!
  - It seems too hard, let us try with trees with NO input

General Picture

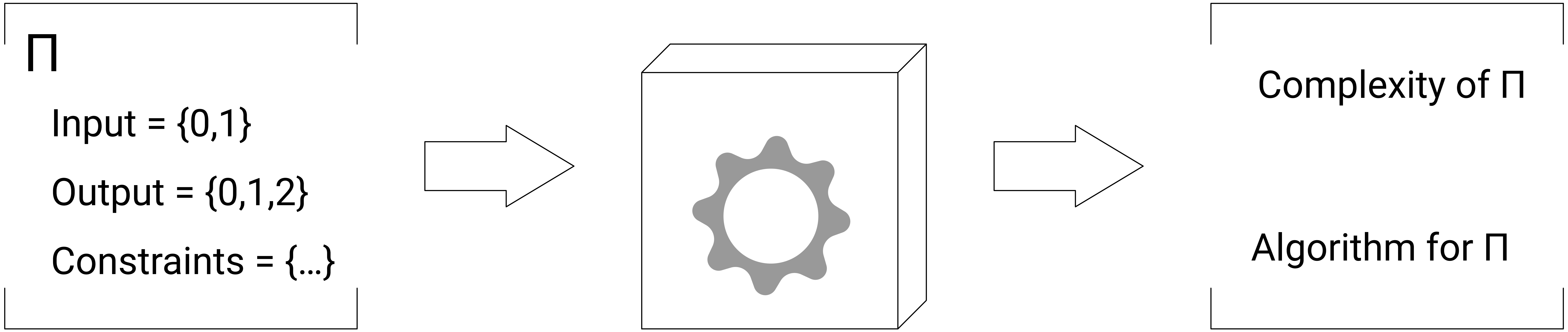
- Grids allow to propagate too much information
- On trees/bounded treewidth graphs it should not be possible
- Let us prove that the complexity of LCLs is decidable on trees!
  - It seems too hard, let us try with trees with NO input
  - The tree structure can be used to encode inputs!

General Picture

- Grids allow to propagate too much information
- On trees/bounded treewidth graphs it should not be possible
- Let us prove that the complexity of LCLs is decidable on trees!
  - It seems too hard, let us try with trees with NO input
  - The tree structure can be used to encode inputs!
  - Let us just try to understand inputs, on cycles

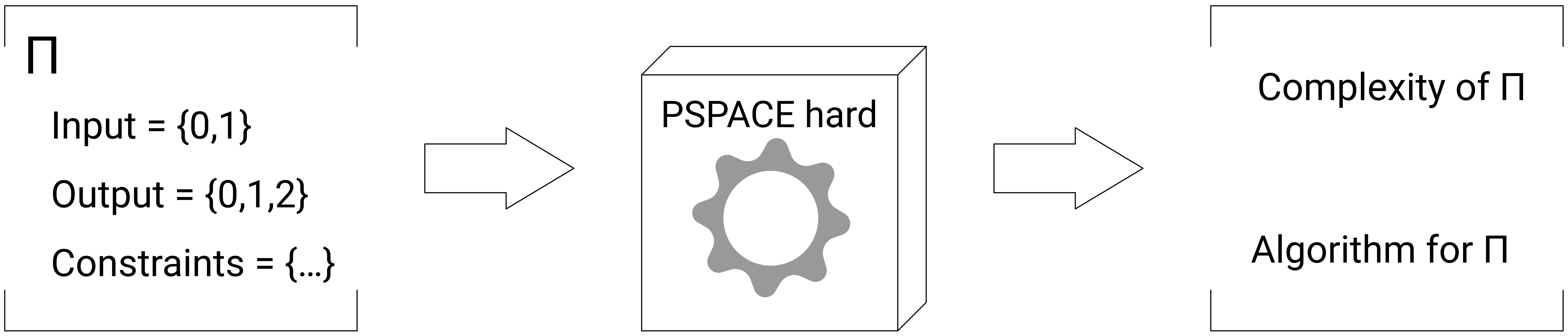
Results

Given an LCL Π on cycles/paths *with input*, it is possible to decide its distributed time complexity, and synthesize an asymptotically optimal algorithm for Π

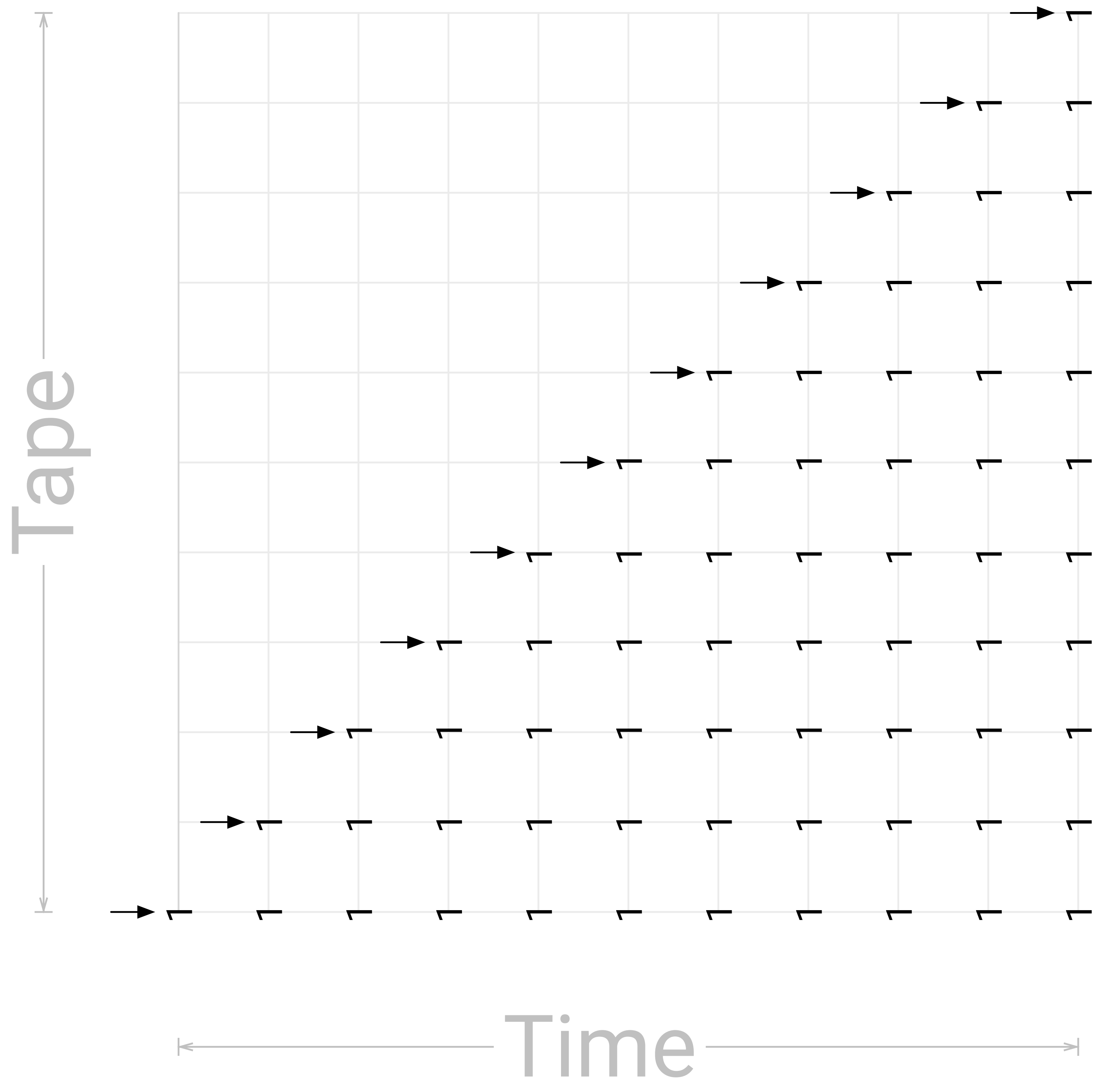


Results

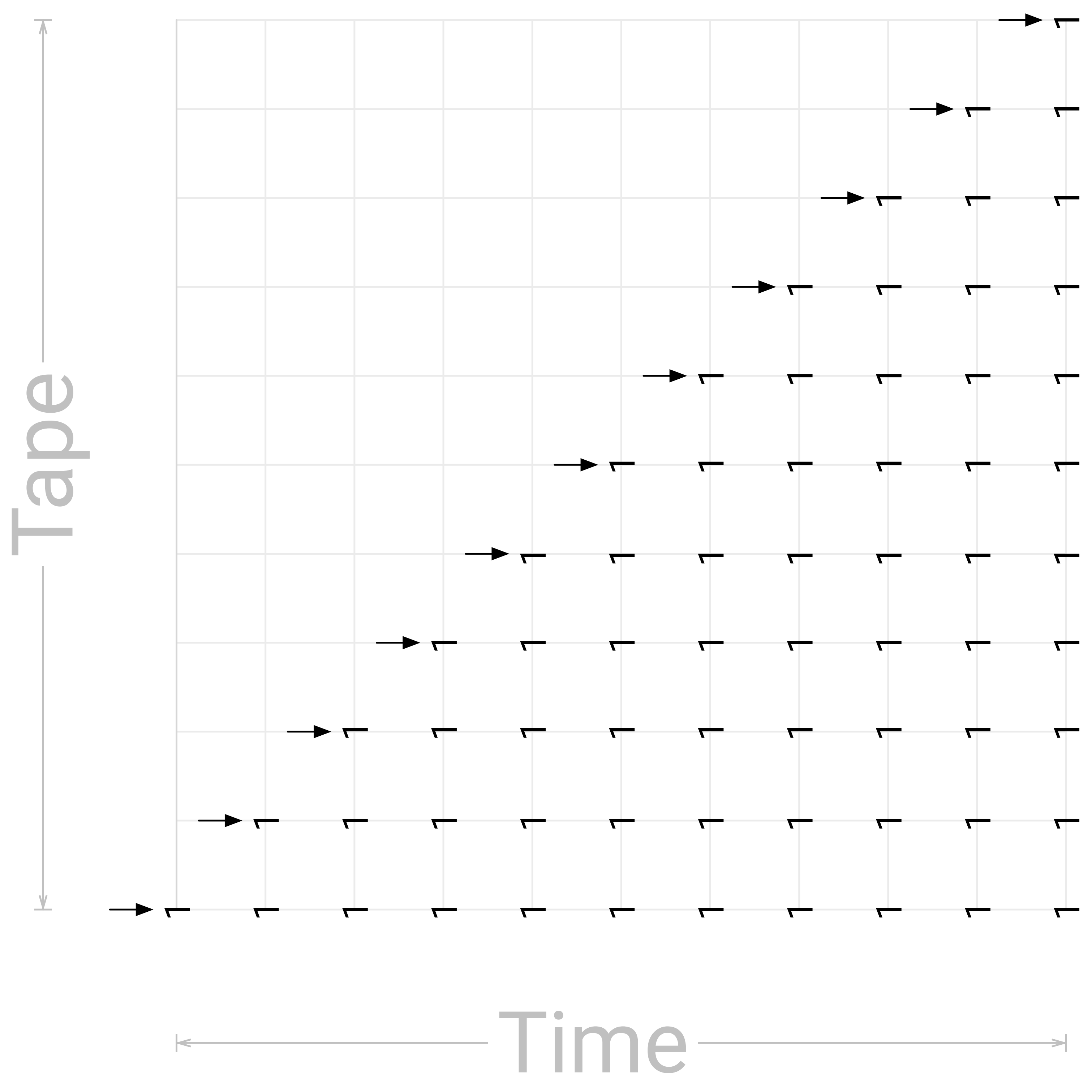
It is PSPACE-hard to distinguish whether an LCL Π on cycles/paths with input labels can be solved in O(1) time or it needs Ω(n) time

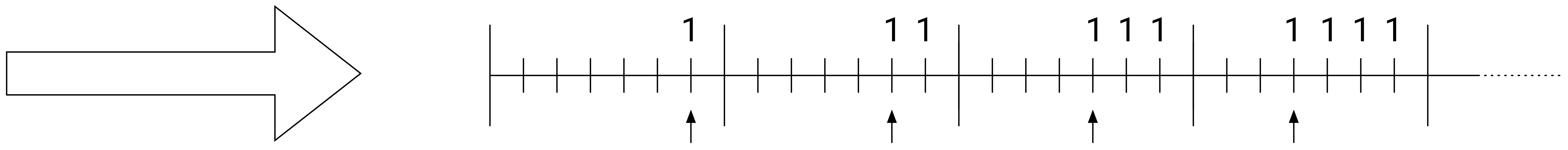


Hardness

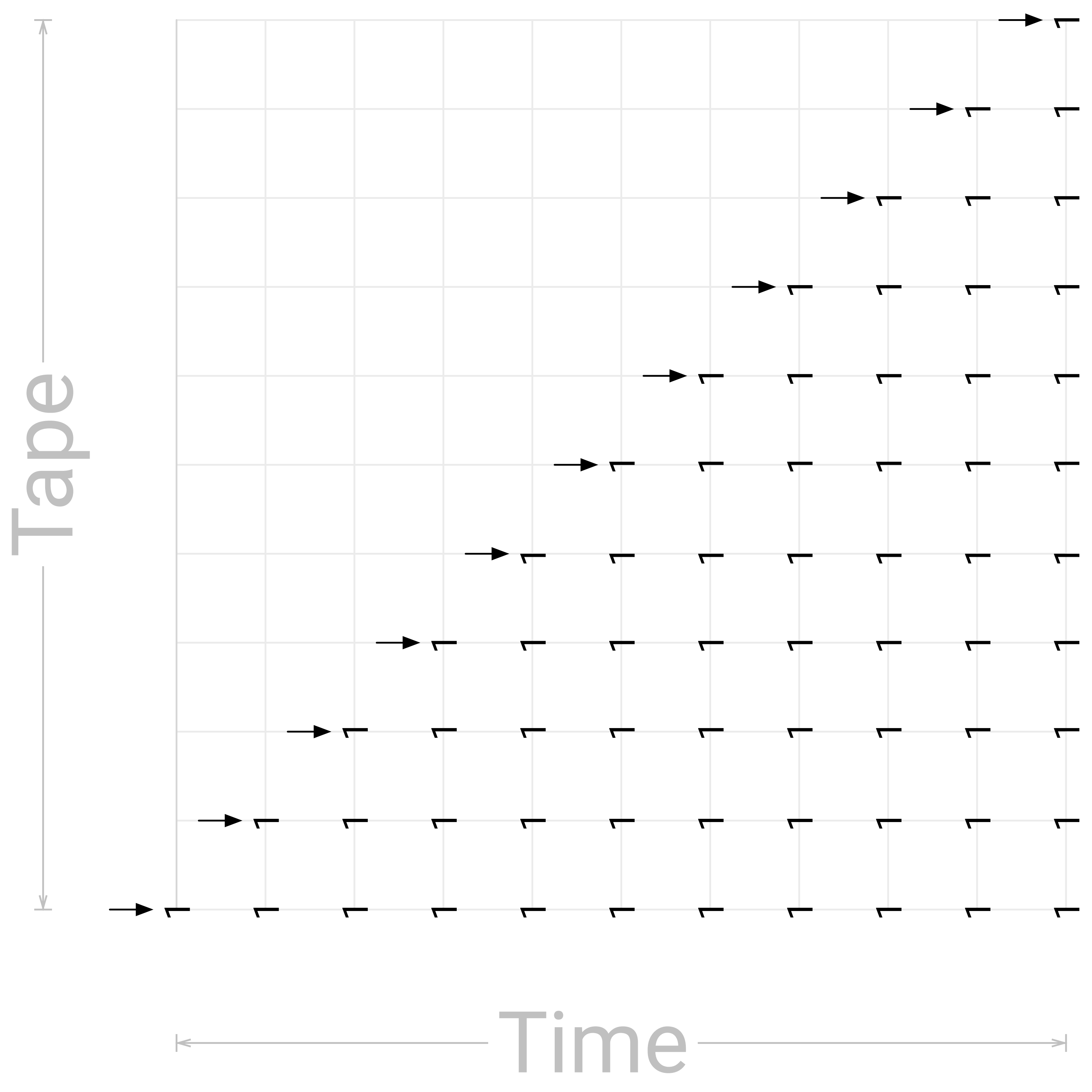


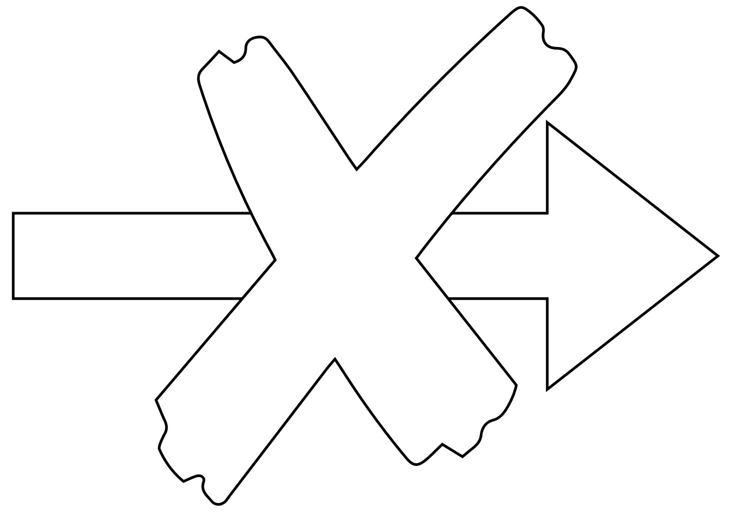
Hardness

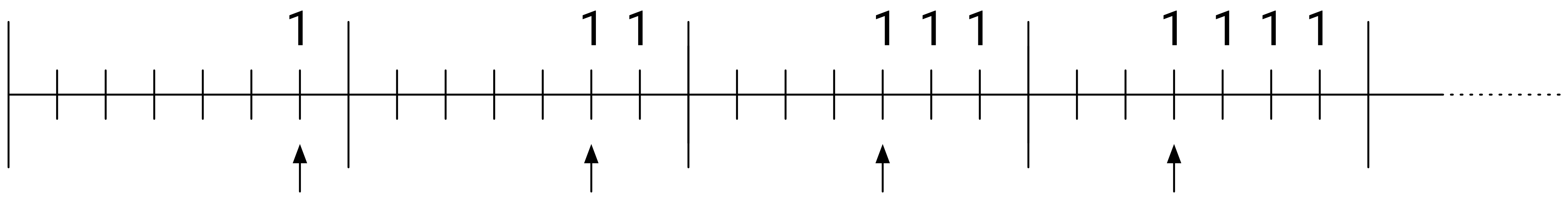




Hardness

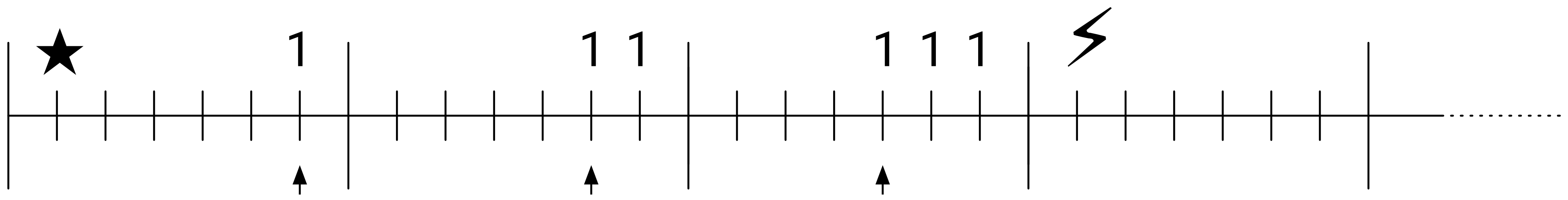






Hardness

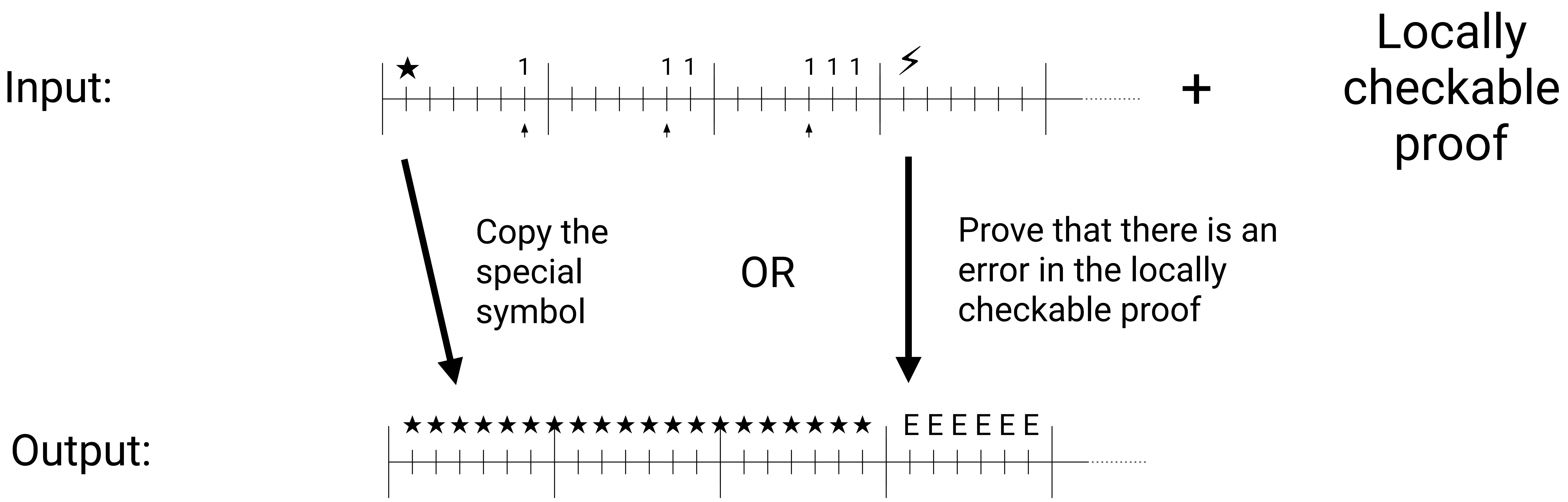
Input:



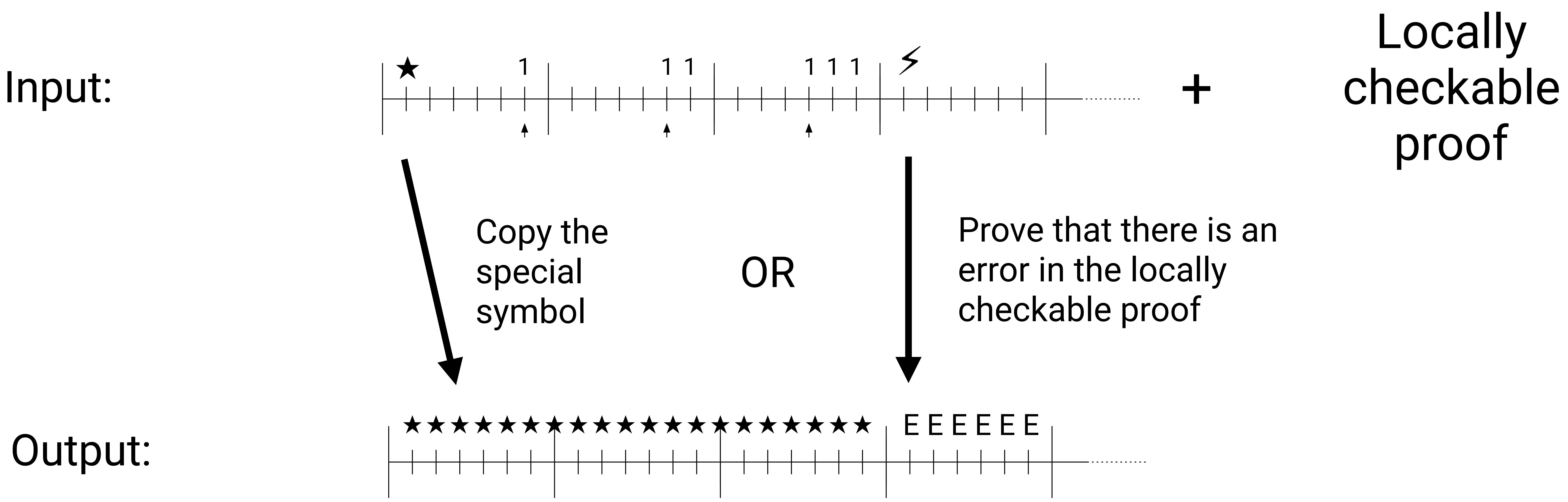
+

Locally checkable proof

Hardness


Input:
★
1
1 1
1 1 1
⚡
+
Locally checkable proof
Copy the special symbol
OR
Prove that there is an error in the locally checkable proof
Output:
★★★★★★★★★★★★★★★★★★★★
E E E E E E

Hardness



The obtained LCL has binary input and it is radius 1 checkable

Conclusions

Conclusions

- We can automatically obtain the complexity of any LCL on paths **with input**

Conclusions

- We can automatically obtain the complexity of any LCL on paths **with input**
- We can automatically obtain an optimal algorithm

Conclusions

- We can automatically obtain the complexity of any LCL on paths **with input**
- We can automatically obtain an optimal algorithm
- What about **trees**?

Conclusions

- We can automatically obtain the complexity of any LCL on paths **with input**
- We can automatically obtain an optimal algorithm
- What about **trees**?
  - Can we decide if an LCL is **$O(\log^* n)$** or **$\Omega(\log n)$**?

Conclusions

- We can automatically obtain the complexity of any LCL on paths **with input**
- We can automatically obtain an optimal algorithm
- What about **trees**?
  - Can we decide if an LCL is **$O(\log^* n)$** or **$\Omega(\log n)$**?
  - What about **regular balanced** trees with **no input**?

Conclusions

- We can automatically obtain the complexity of any LCL on paths **with input**
- We can automatically obtain an optimal algorithm
- What about **trees**?
  - Can we decide if an LCL is **$O(\log^* n)$** or **$\Omega(\log n)$**?
  - What about **regular balanced** trees with **no input**?

Thank you!