The Landscape of Distributed Time Complexity

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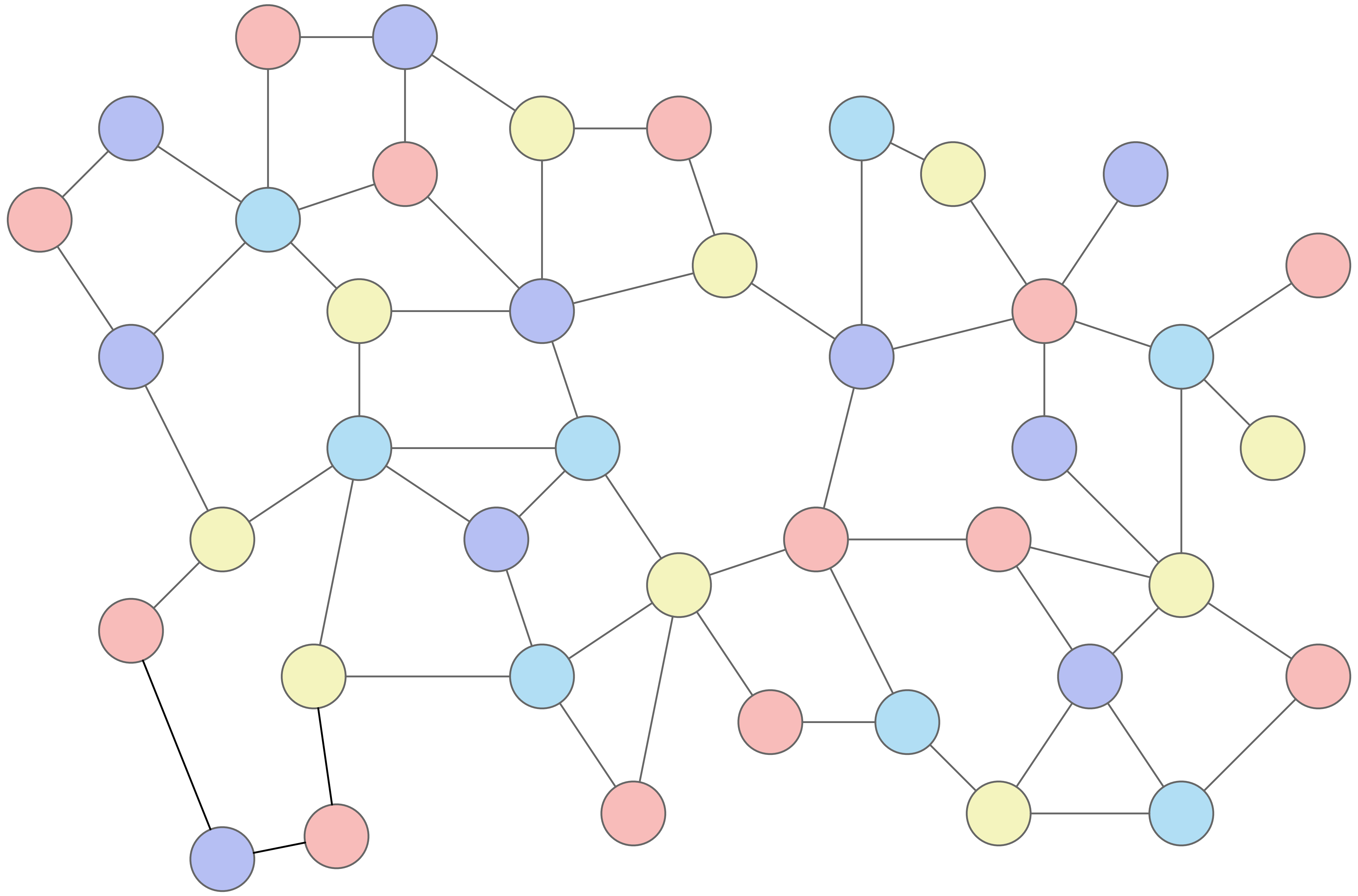
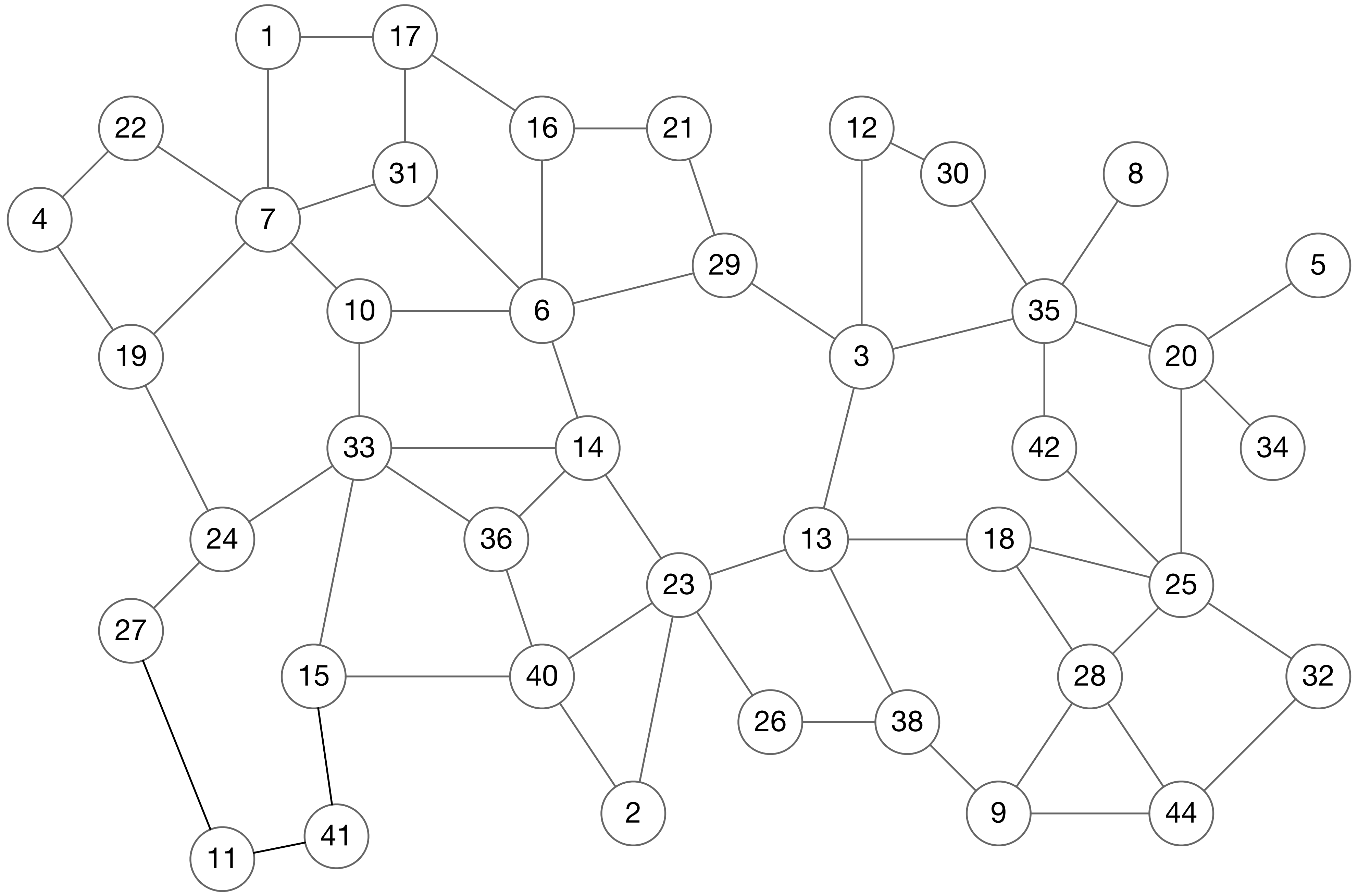
Topic: distributed graph problems

- Family of graph problems: ***LCLs***
- Focus on ***locality***
  - How much does an entity need to know about the graph in order to solve a graph problem?
  - How local can these problems be?
  - When can randomness help?



LOCAL model

- ***Graph*** = communication network
- ***Synchronous*** rounds
- Time complexity = ***number of rounds*** required to solve the problem
- Nodes have ***IDs***
- ***No bounds*** on the computational power of the entities and on the bandwidth

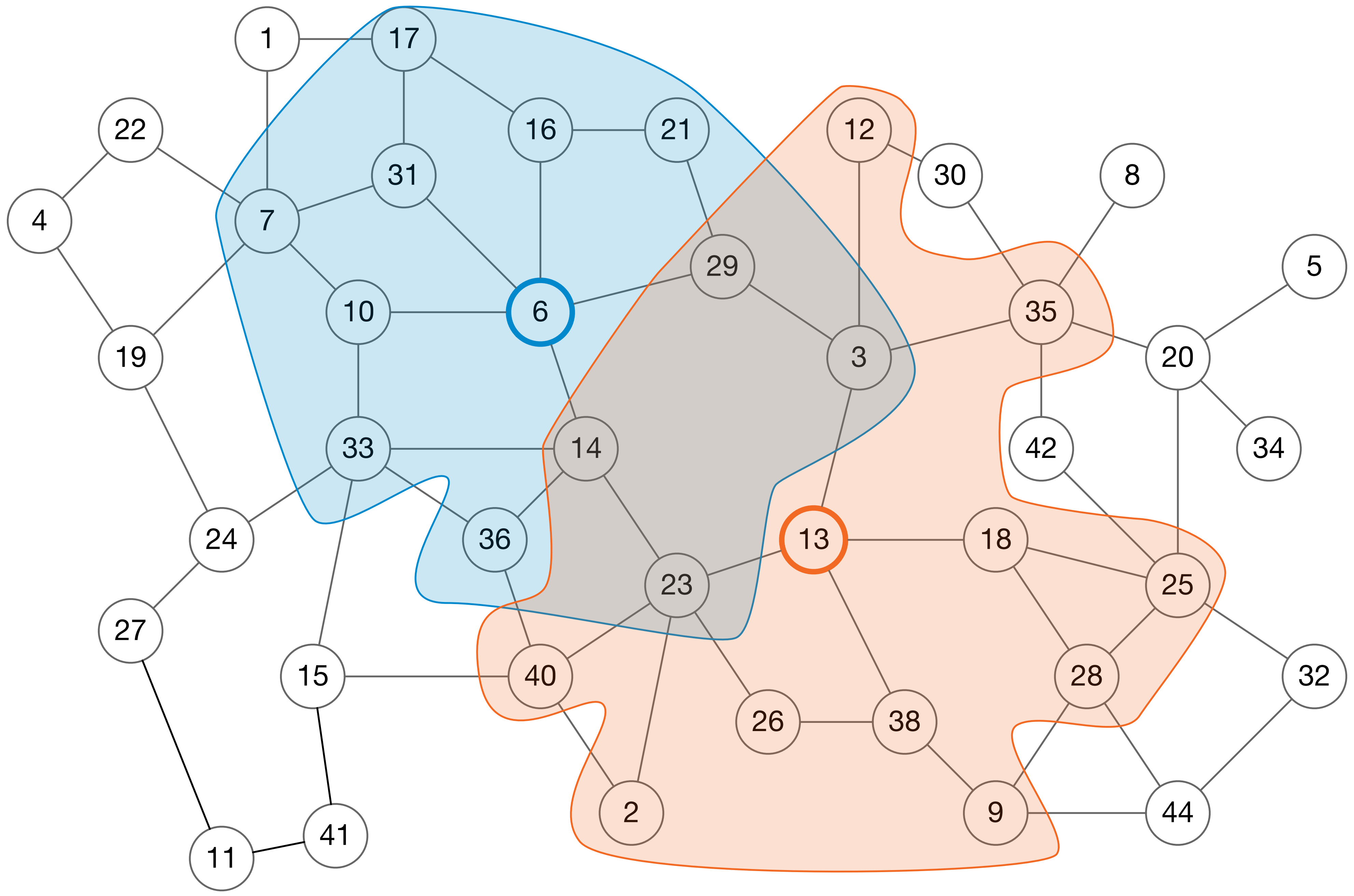


LOCAL model

- **Initial knowledge** of a node:
  - ***n*** = the total number of nodes in the graph
  - Δ = the maximum degree of the graph
  - Its unique ***ID***
  - A ***port numbering*** of its incident edges
  - Sequence of ***random bits***

LOCAL model

- After ***t* rounds**:
  - knowledge of the graph up to **distance *t***
- Focus on **locality**:
  - time = number of rounds = distance



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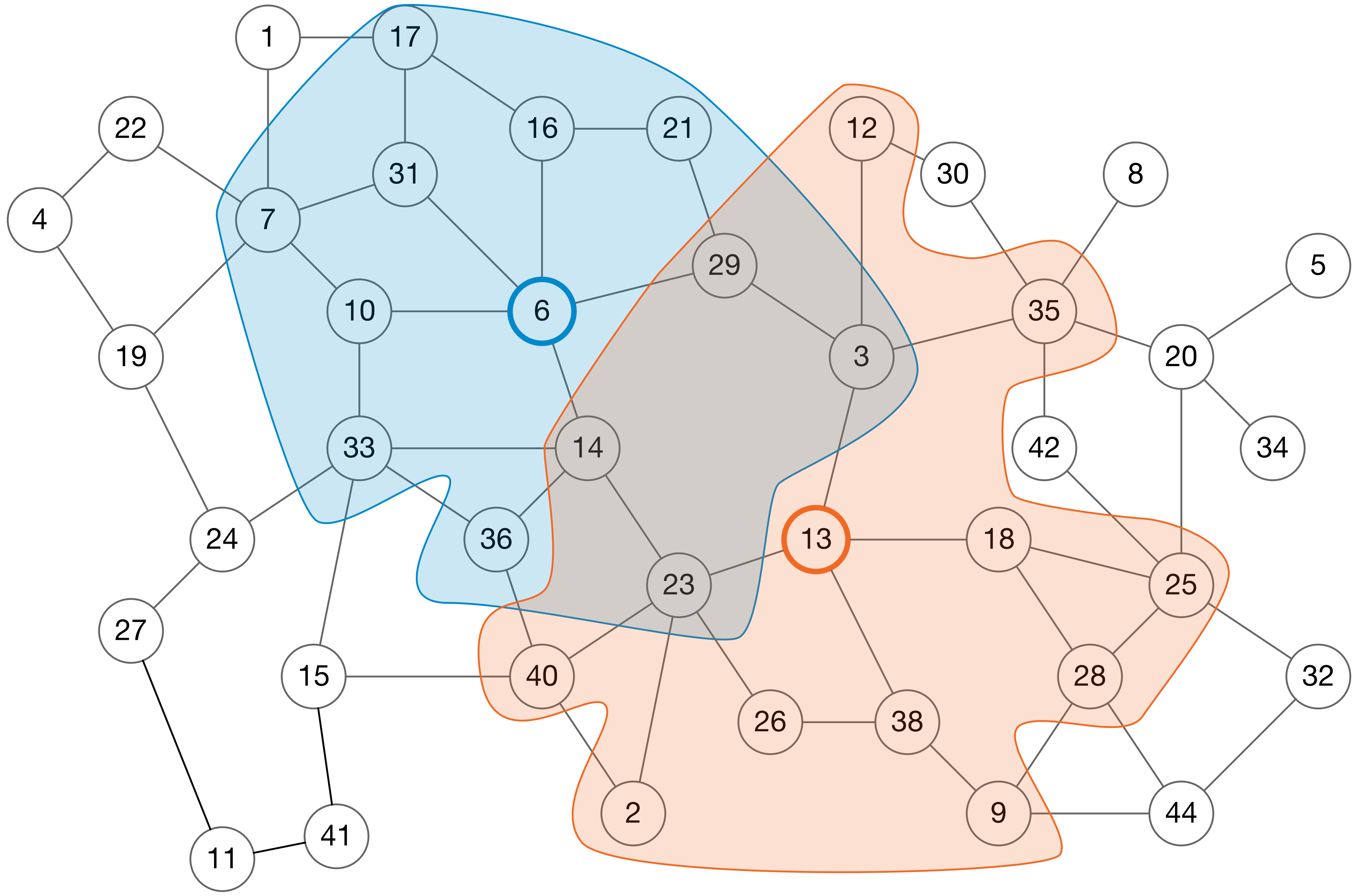

***Everything** can be solved in **Diameter time**!*

Locally Checkable Labelings (LCLs)

A **family of graph problems** that includes many important problems

Maximal Independent Set, Maximal Matching, vertex coloring, edge coloring…

Locally Checkable Labelings (LCLs)

- **Input**
    - Graph of ***constant*** maximum degree Δ
    - Node labels from a ***constant-size*** set X

[Naor and Stockmeyer, 1995]

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- **Output**
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- **Correctness**
  - A solution is globally correct if it is correct in ***all constant-radius*** neighborhoods

[Naor and Stockmeyer, 1995]

Locally Checkable Labelings (LCLs)

Two algorithms:

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- ***Prover***: runs for some time and produces an output, plus a certificate of correctness

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Two algorithms:

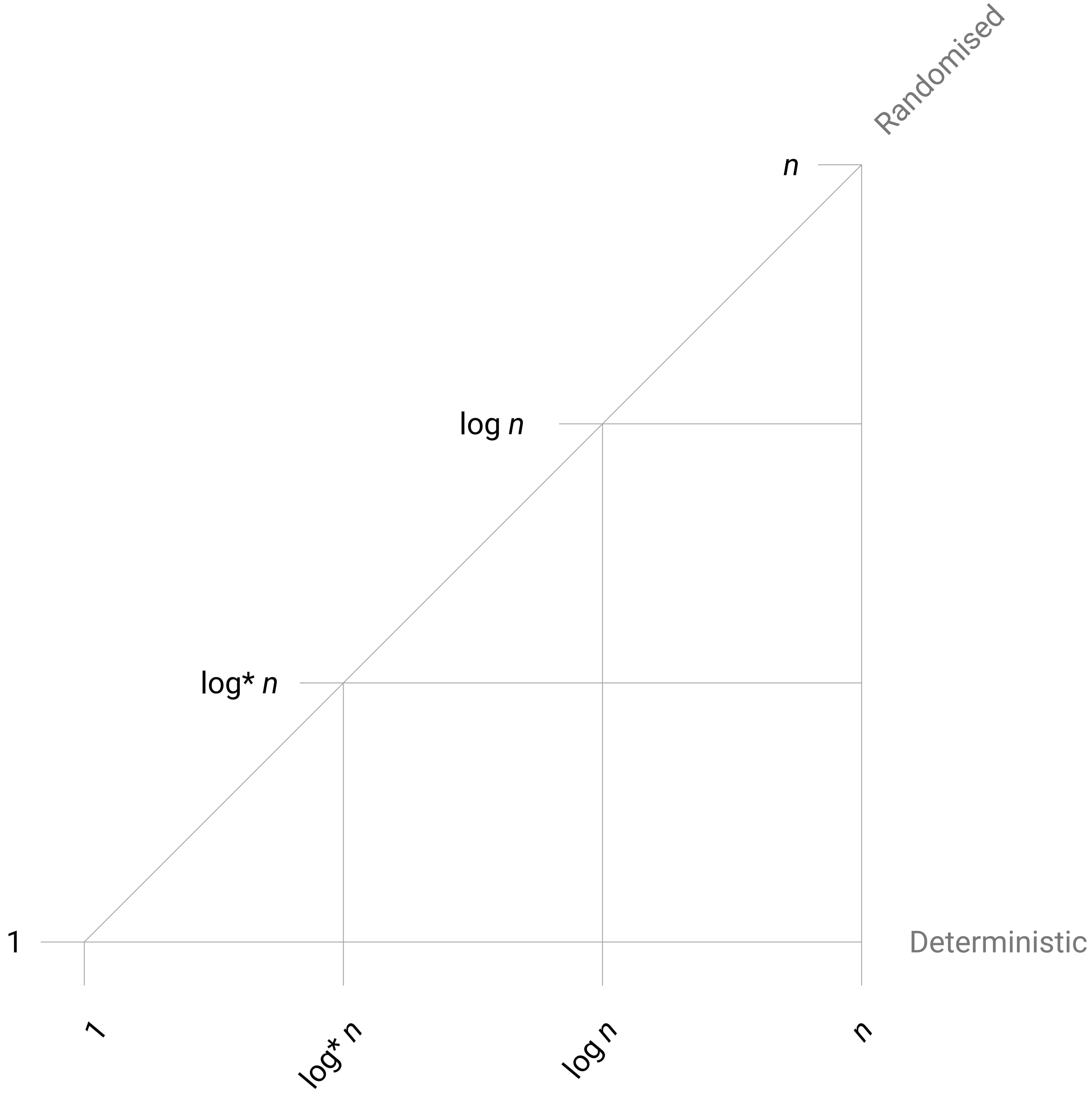
- ***Prover***: runs for some time and produces an output, plus a certificate of correctness
- ***Verifier***: checks, in constant time, if the certificate is correct

Example: weak 2-coloring

- **Output**: color nodes from a palette of **2 colors**
- **Constraint**: each node must have a **different color** from **at least 1** neighbor

Landscape of LCLs

- **Which time complexities** are possible for **LCLs**?
- How **local** are **LCLs**?
- **Does randomness help** in solving an **LCL** faster?


Randomised
n
log n
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Deterministic
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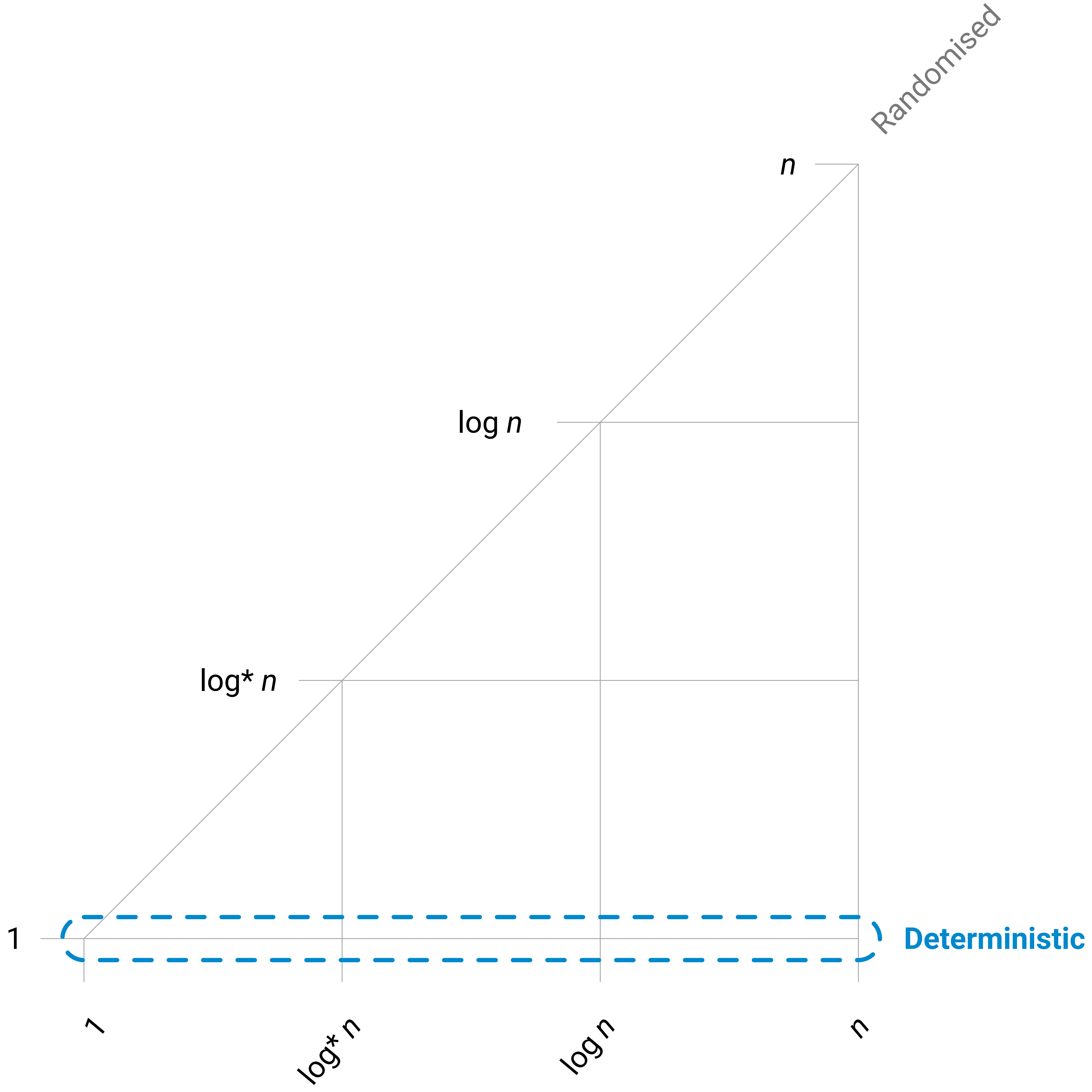
log* n
log n
n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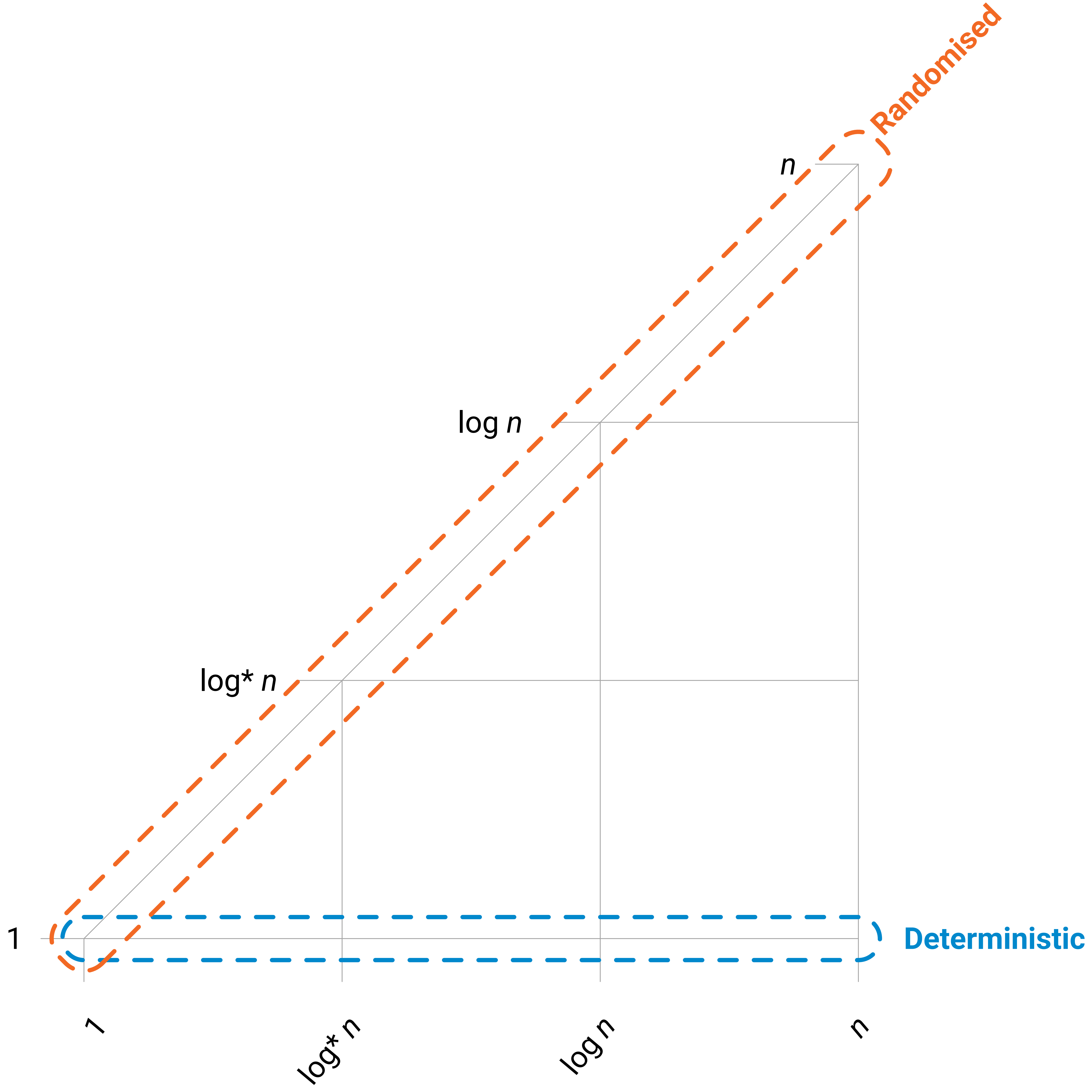

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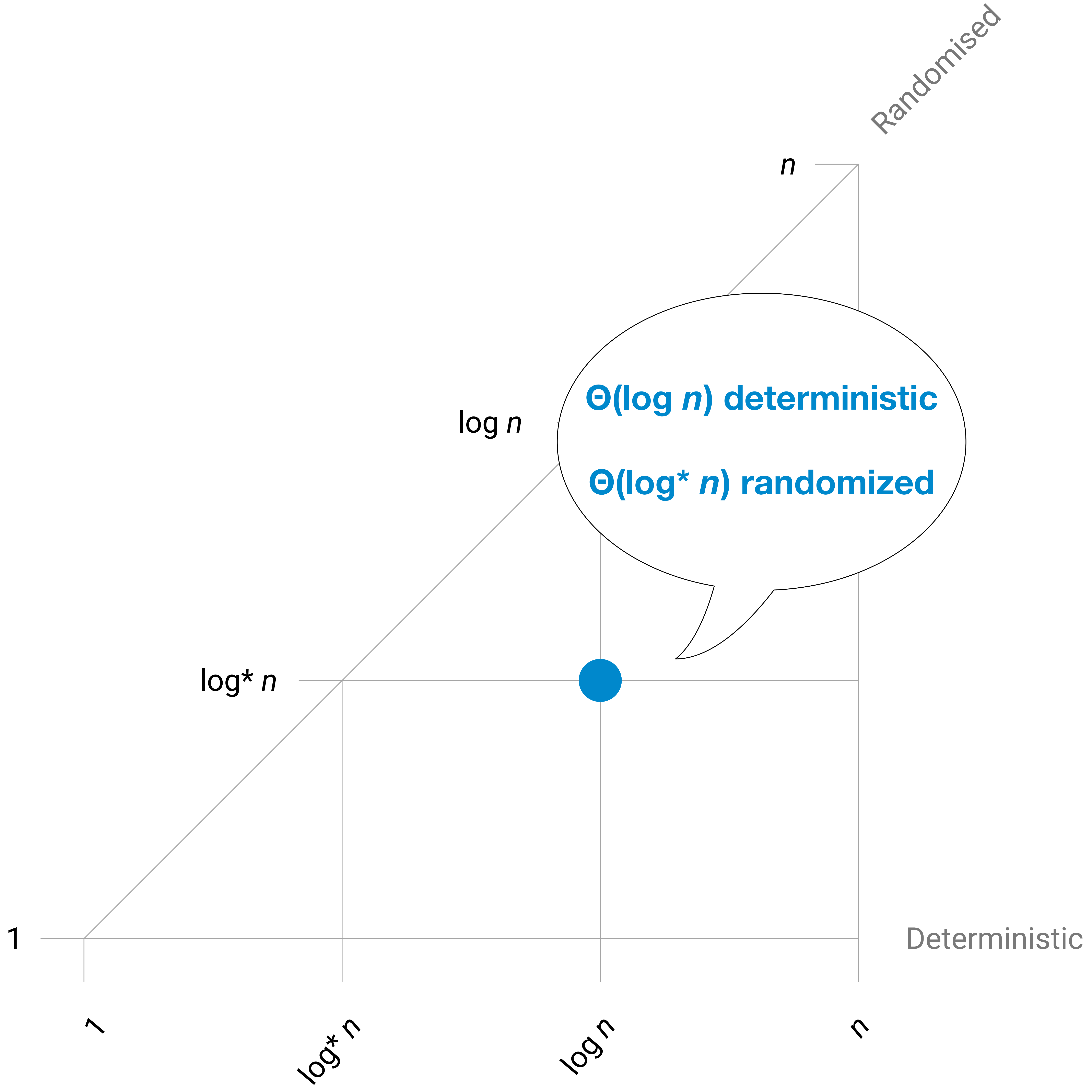
Randomised
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Θ(log n) deterministic
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Deterministic
1
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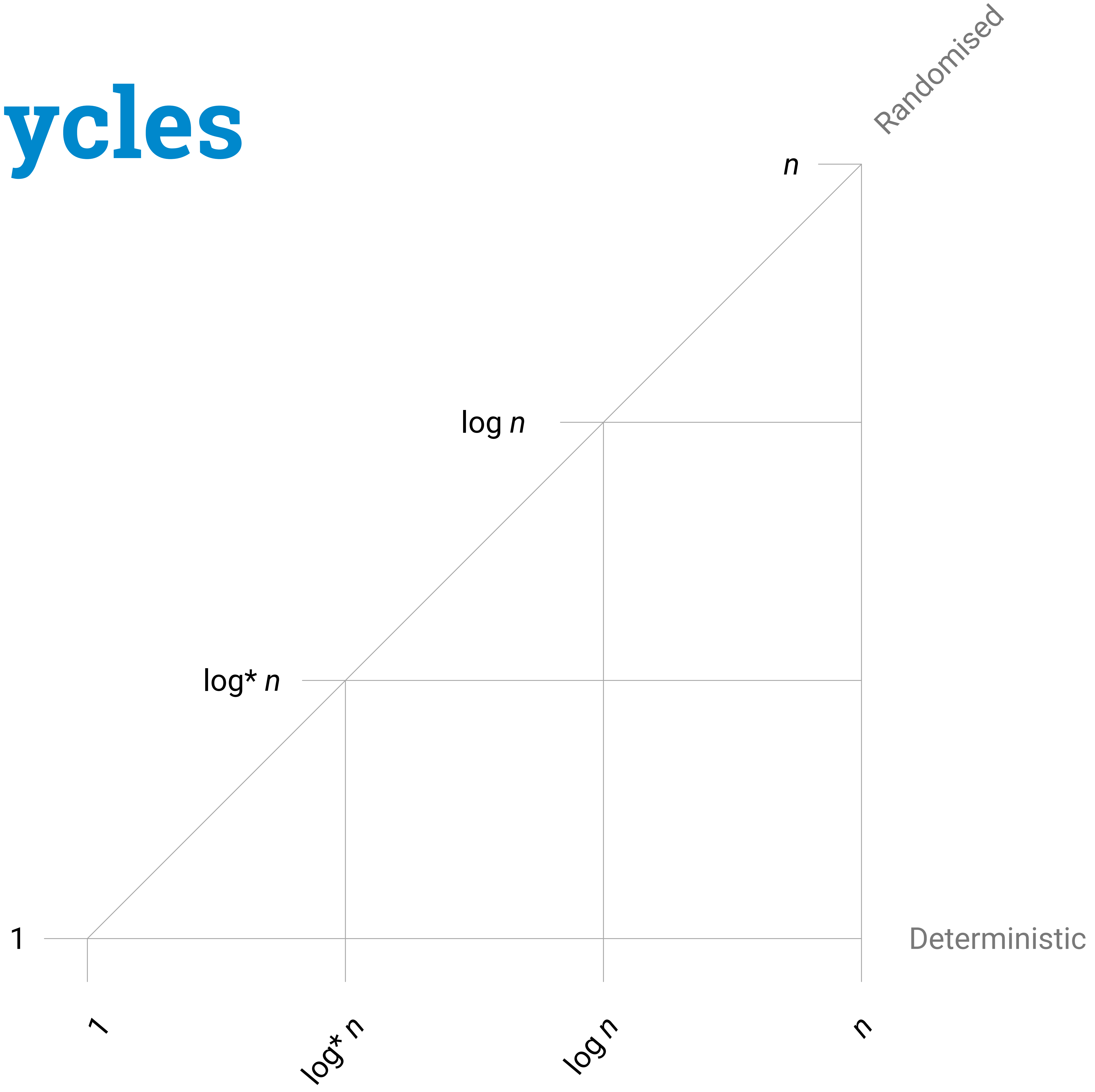
Paths/Cycles



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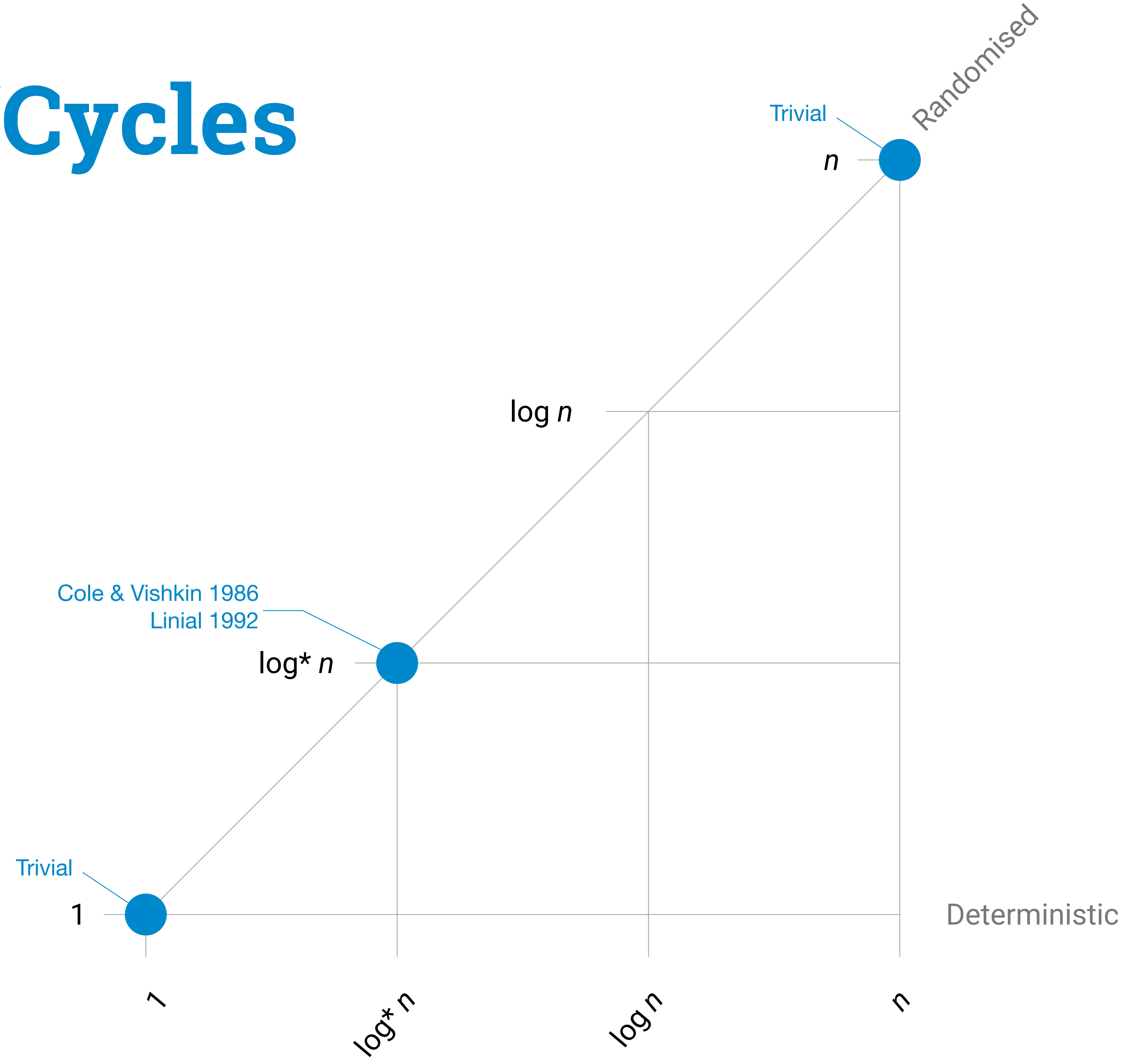

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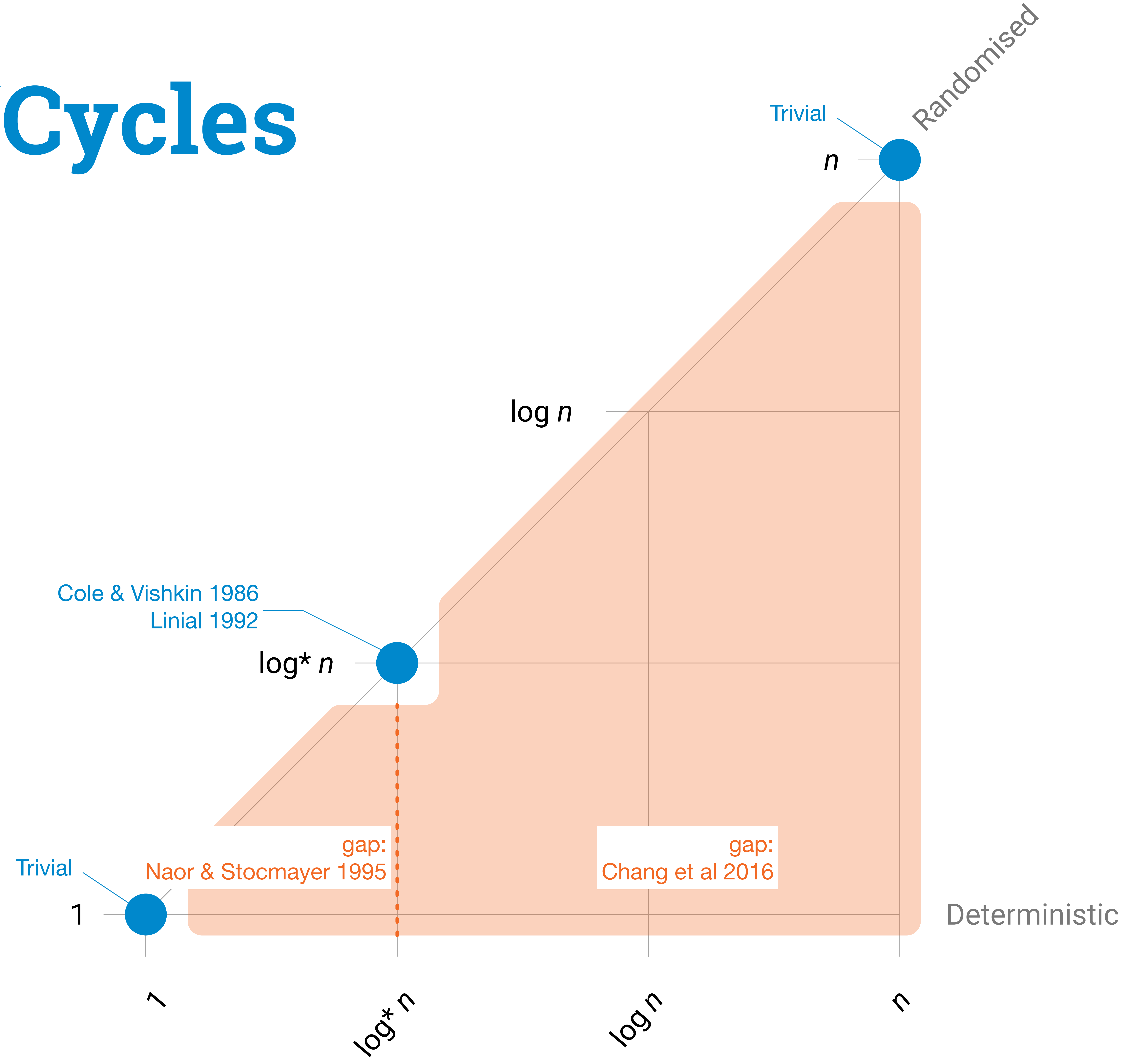

Gaps

- **$\omega(1)$ — $o(\log^* n)$ gap**:
  - Every **algorithm A** that solves an **LCL P** in **$o(\log^* n)$** rounds can be **automatically sped up** into an **algorithm A'** that solves **P** in **$O(1)$** rounds
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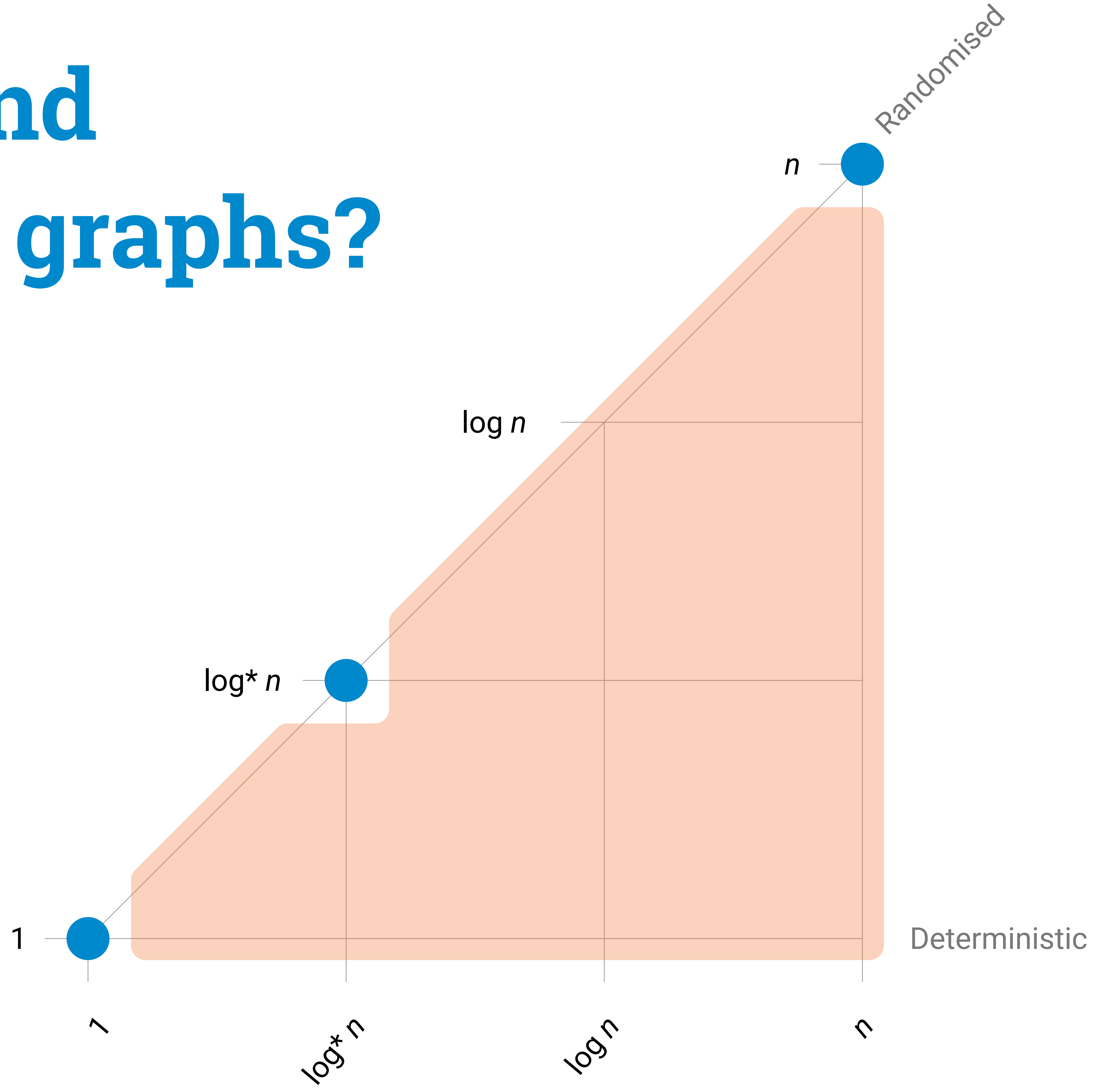
  - Every **algorithm A** that solves an **LCL P** in **$o(n)$** rounds can be **automatically sped up** into an algorithm **A'** that solves **P** in **$O(\log^* n)$** rounds

Using gaps for proving lower bounds

- The ***$\omega(\log^* n) - o(n)$*** gap gives also a ***normalized*** way to solve LCLs:
  - Find a distance-k coloring in ***$O(\log^* n)$*** time
  - Use the coloring in ***constant*** time
- **Easy** way to prove an ***$\Omega(n)$*** lower bound:
  - Prove that, for any k, a problem can not be solved in O(1) rounds given a distance-k coloring!

Trees and general graphs?



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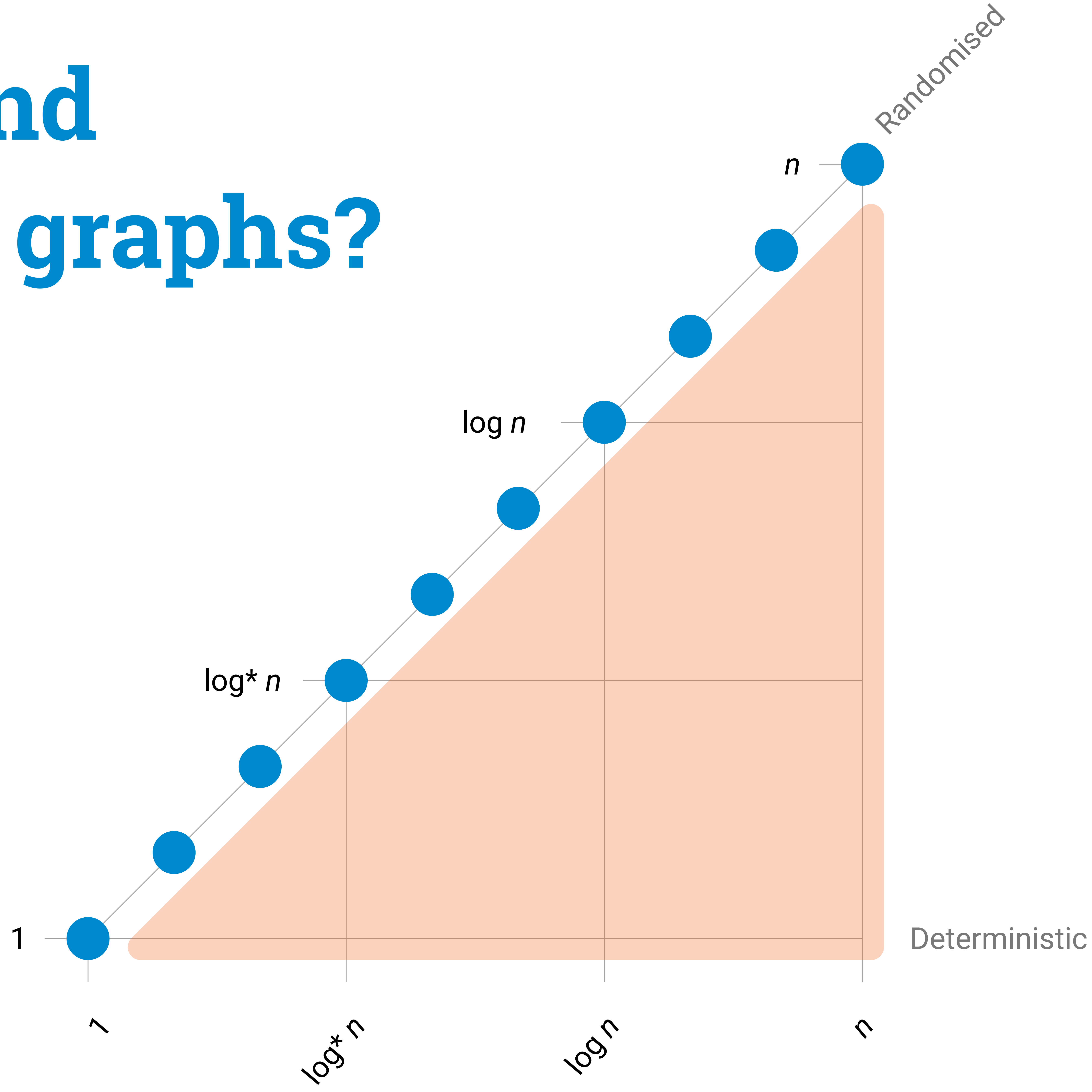

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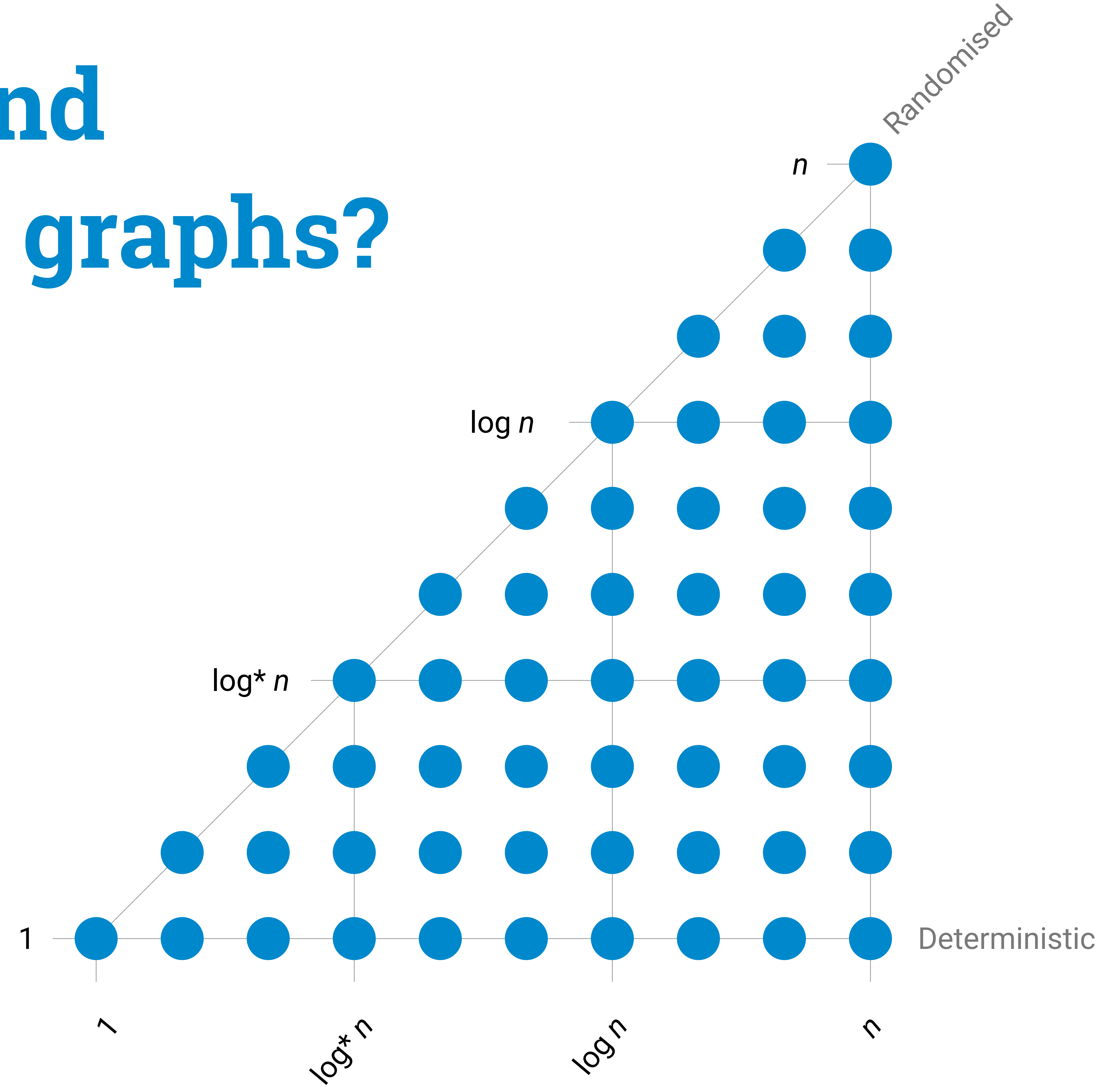

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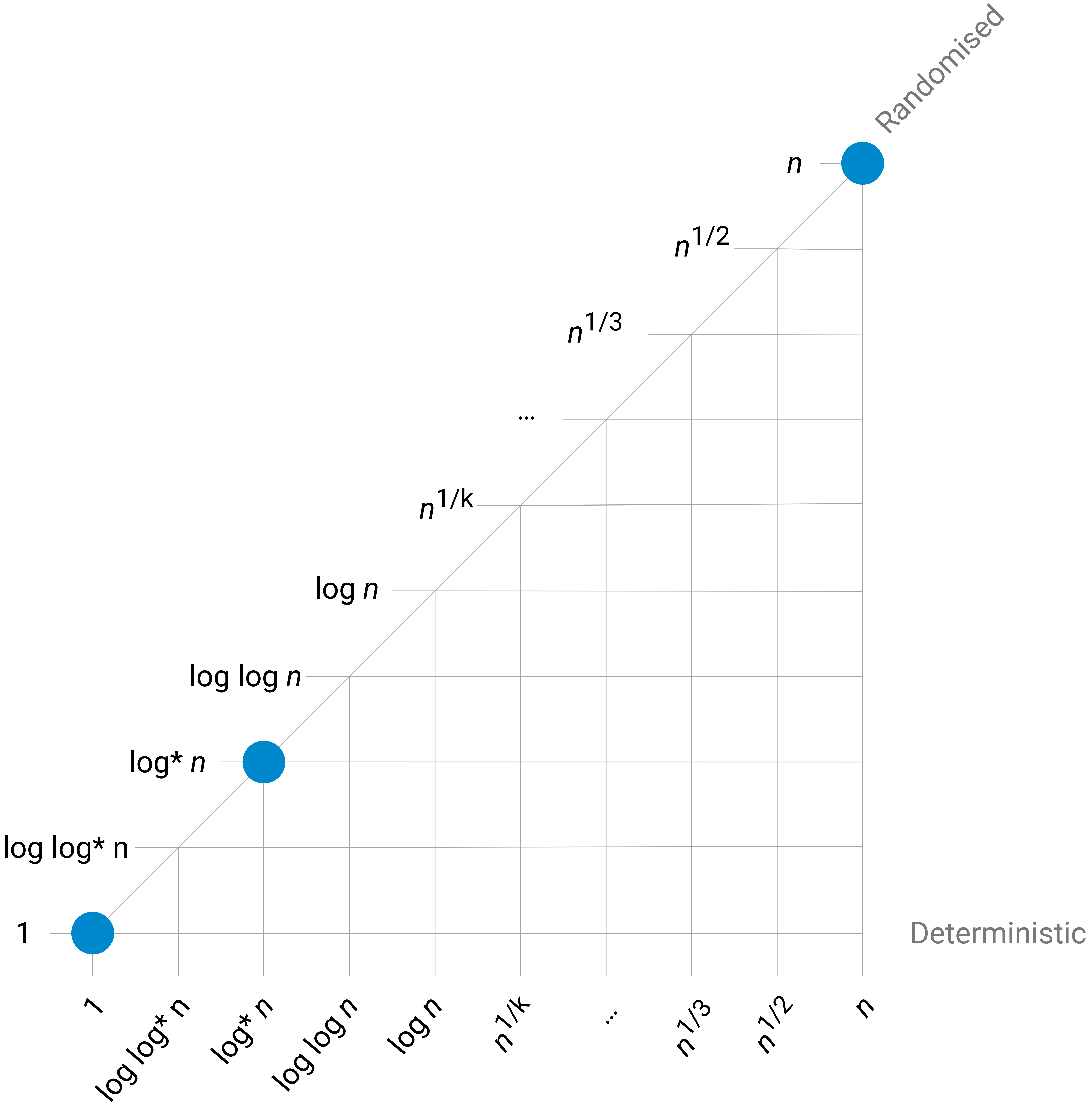
Lots of progress since 2016

- Brandt, Fischer, Hirvonen, Keller, Lempiäinen, Rybicki, Suomela, Uitto **[STOC 2016]**
- Chang, Kopelowitz, Pettie **[FOCS 2016]**
- Ghaffari, Su **[SODA 2017]**
- Brandt, Hirvonen, Korhonen, Lempiäinen, Östergård, Purcell, Rybicki, Suomela, Uznański **[PODC 2017]**
- Fischer, Ghaffari **[DISC 2017]**
- Chang, Pettie **[FOCS 2017]**
- Chang, He, Li, Pettie, Uitto **[SODA 2018]**
- Balliu, Hirvonen, Korhonen, Lempiäinen, O., Suomela **[STOC 2018]**
- Ghaffari, Hirvonen, Kuhn, Maus **[PODC 2018]**
- Balliu, Brandt, O., Suomela **[DISC 2018]**
- Balliu, Brandt, O., Suomela **[Unpublished 2019]**

Trees


Randomised
n
$n^{1/2}$
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...
$n^{1/k}$
log n
log log n
log* n
log log* n
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Deterministic
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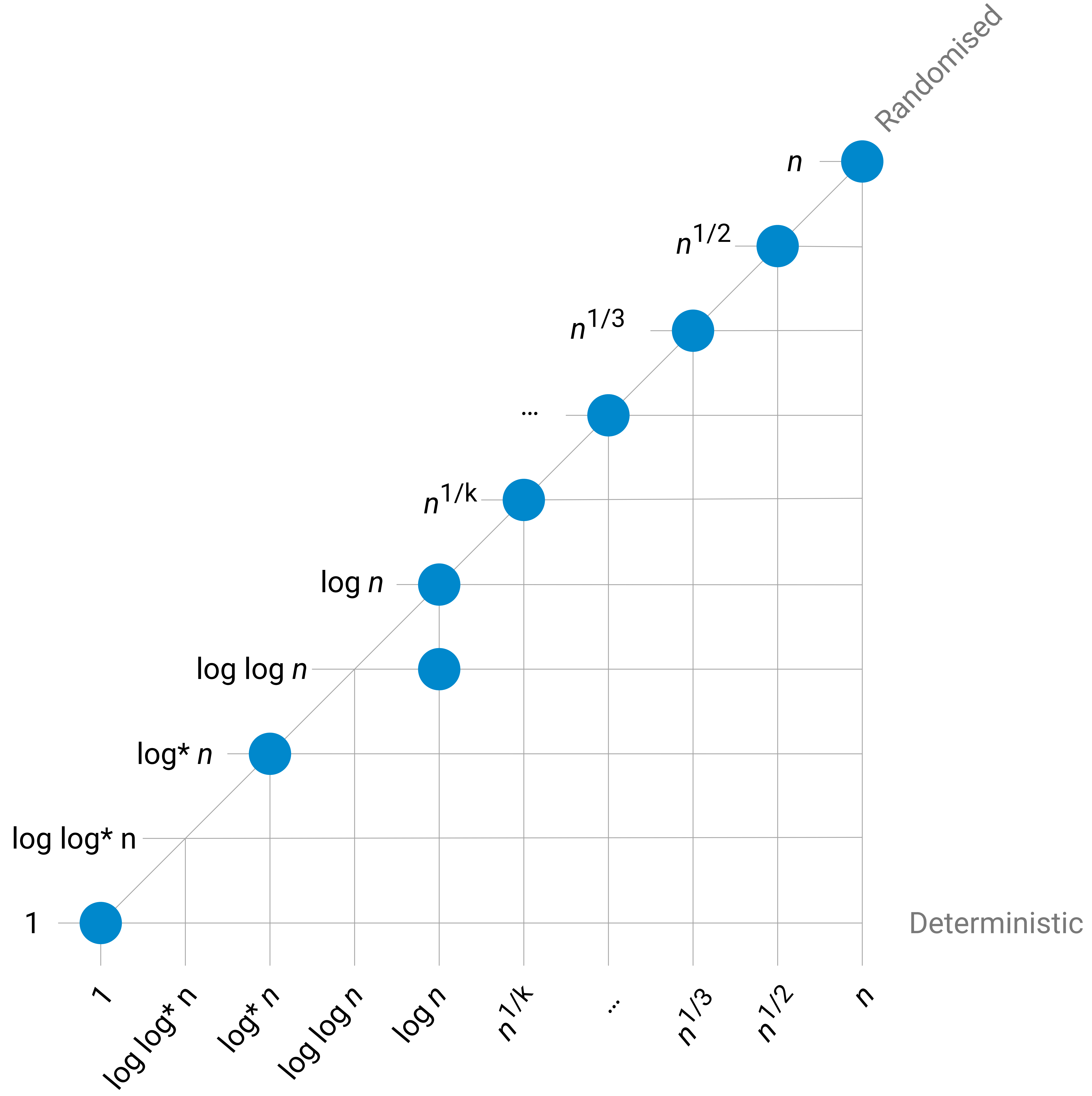
log log* n
log* n
log log n
log n
$n^{1/k}$
...
$n^{1/3}$
$n^{1/2}$
n

Trees



Randomised
n
n1/2
n1/3
...
n1/k
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log log n
log* n
log log* n
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Deterministic
1
log log* n
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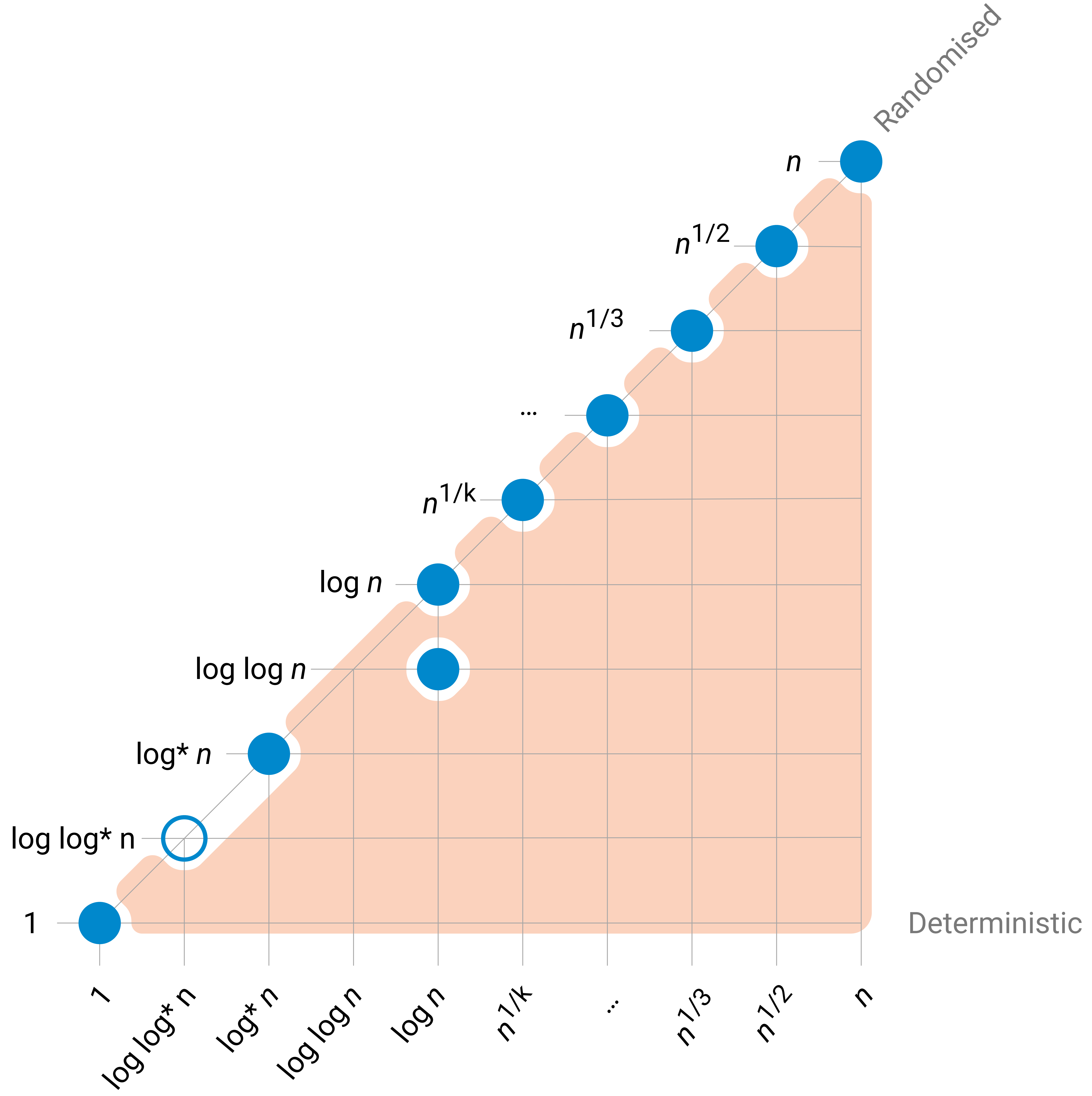
log log n
log n
n 1/k
...
n 1/3
n 1/2
n

Trees



Randomised
n
n1/2
n1/3
...
n1/k
log n
log log n
log* n
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Deterministic
1
log log* n
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log log n
log n
n1/k
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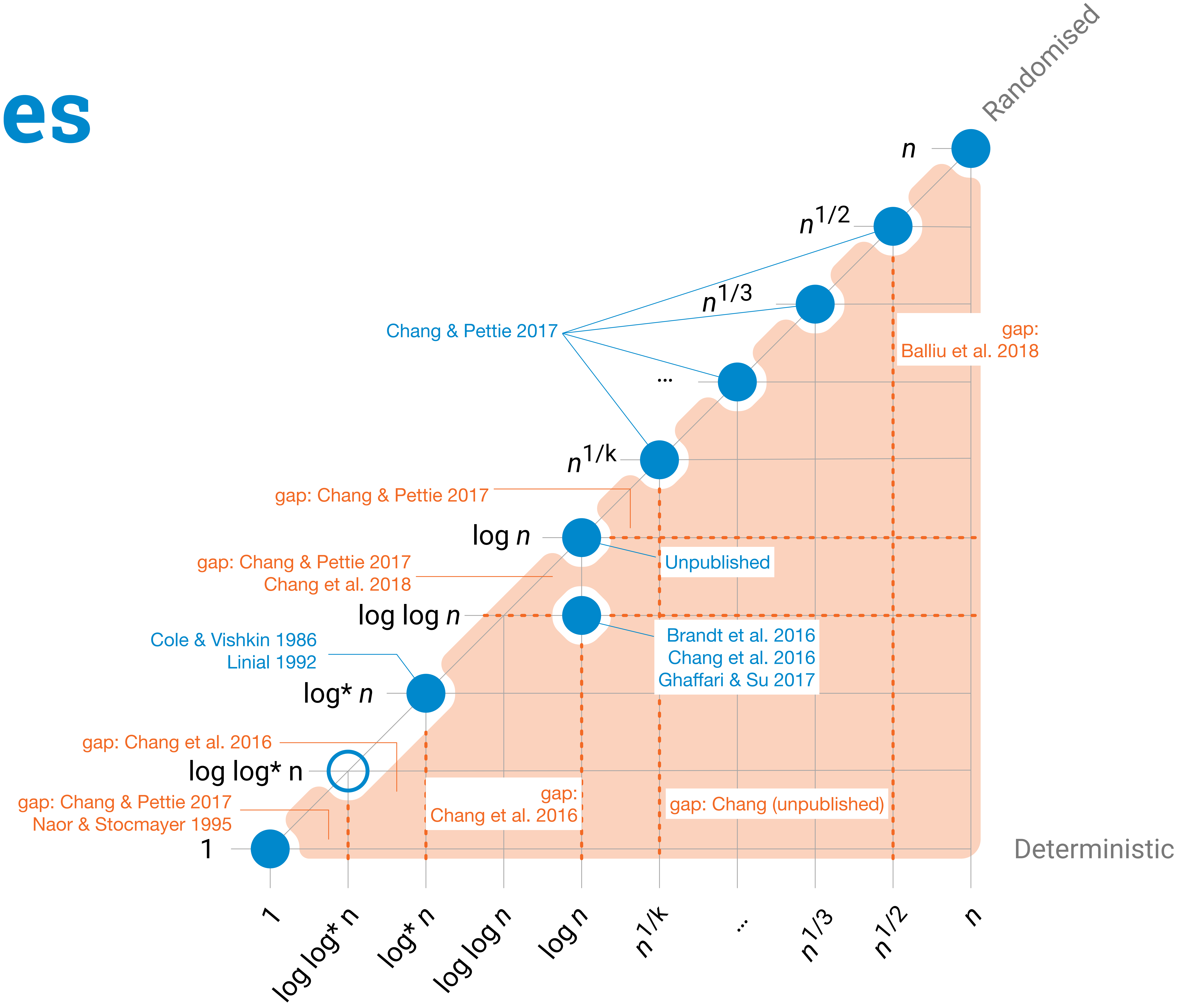
n1/3
n1/2
n

Trees



Randomised
Deterministic
n
$n^{1/2}$
$n^{1/3}$
...
$n^{1/k}$
log n
log log n
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1
Chang & Pettie 2017
gap: Balliu et al. 2018
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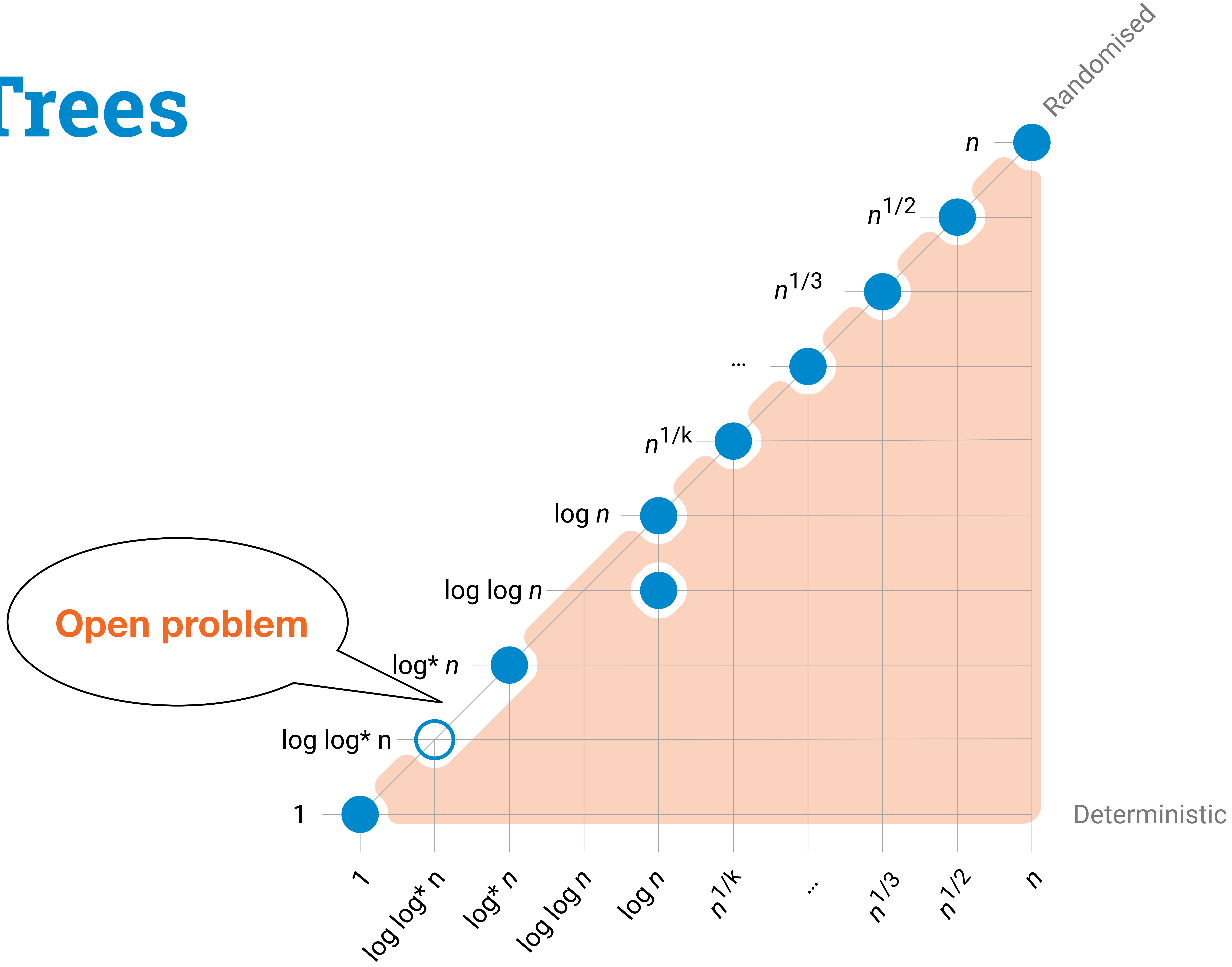
1
log log* n
log* n
log log n
log n
$n^{1/k}$
...
$n^{1/3}$
$n^{1/2}$
n

Trees


Randomised
Deterministic
Open problem
n
n1/2
n1/3
...
n1/k
log n
log log n
log* n
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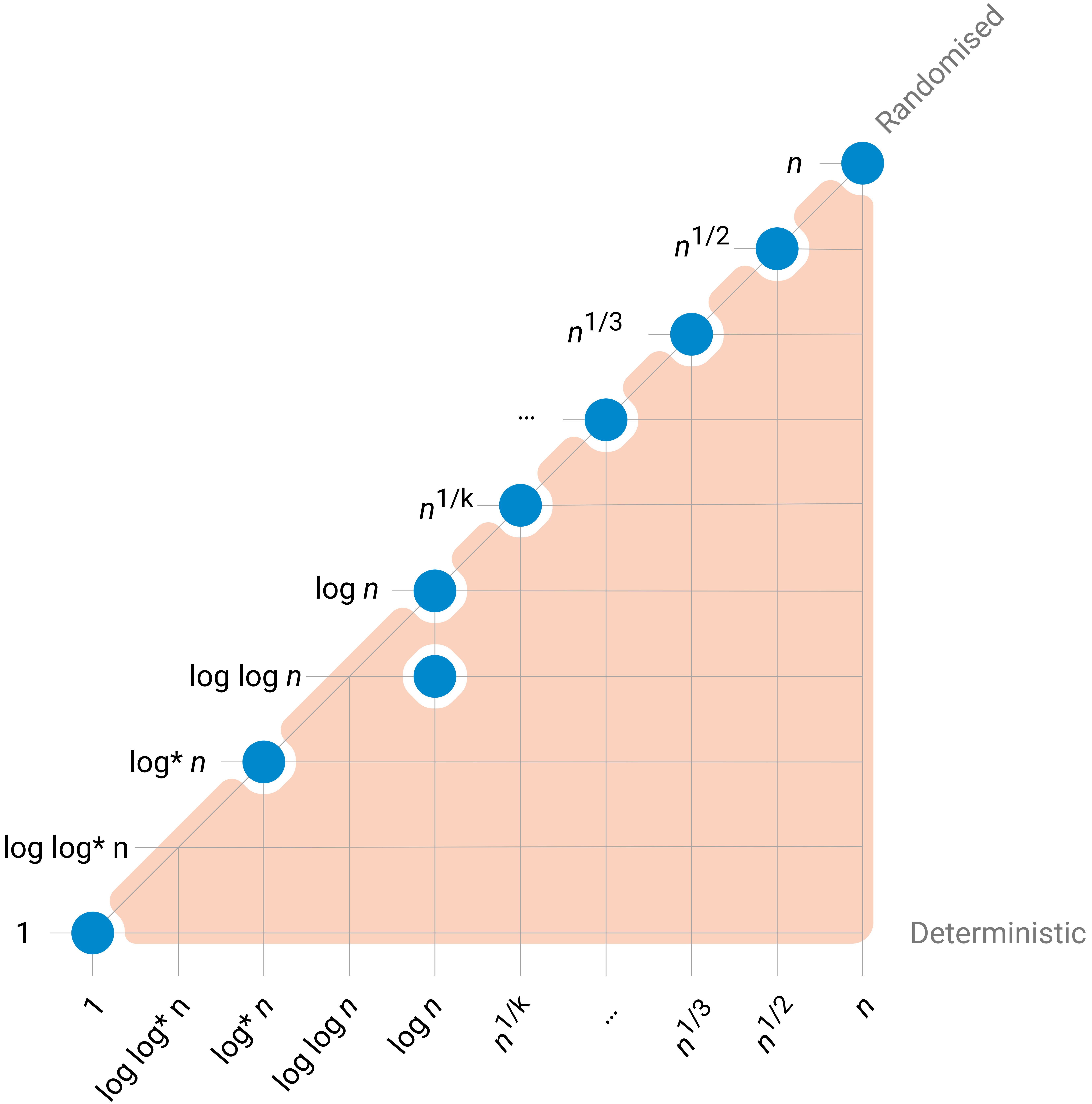
log log* n
log* n
log log n
log n
n 1/k
...
n 1/3
n 1/2
n

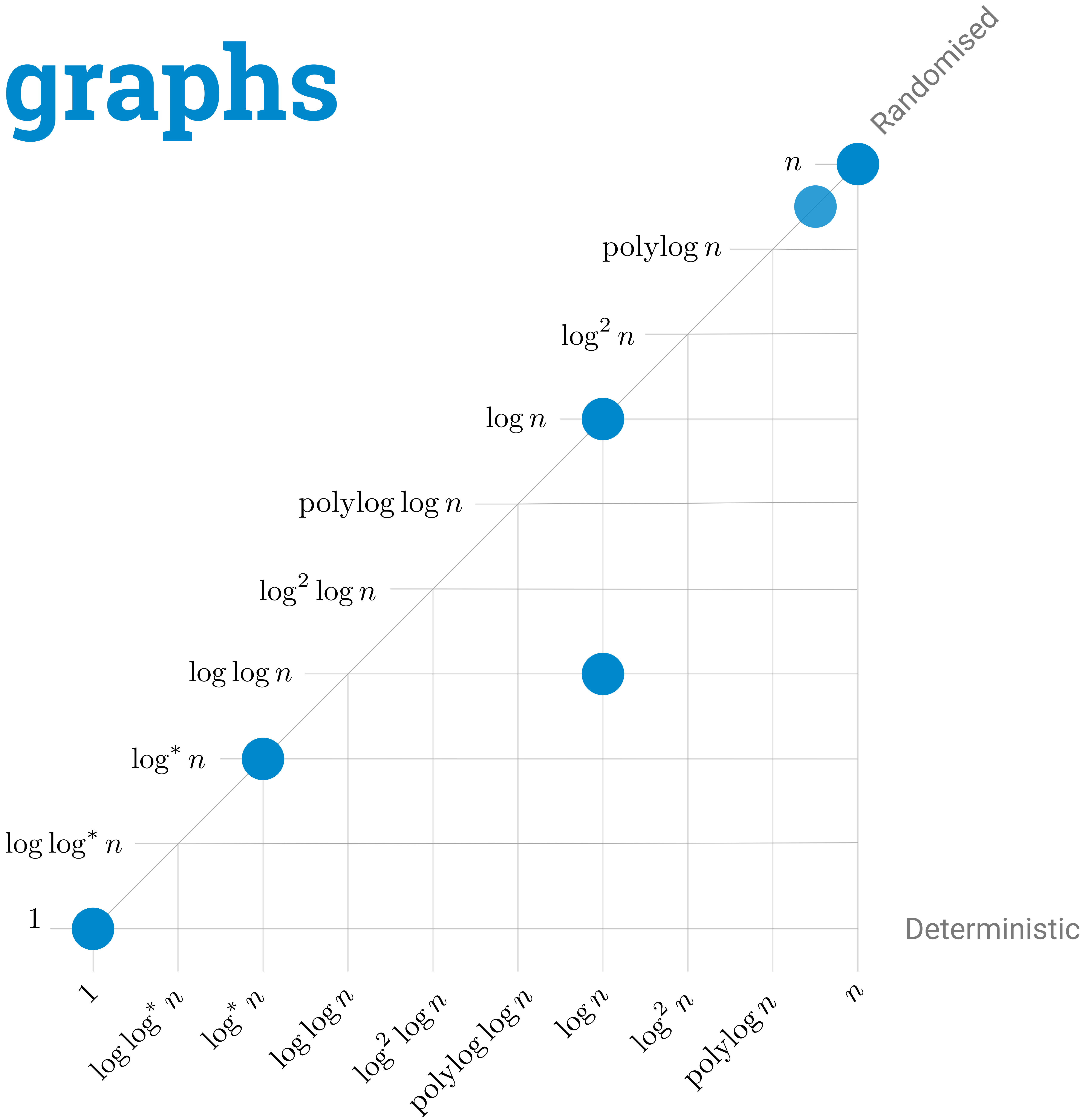
Trees



Homogeneous LCLs

[Balliu, Hirvonen, O., Suomela 2019]

General graphs



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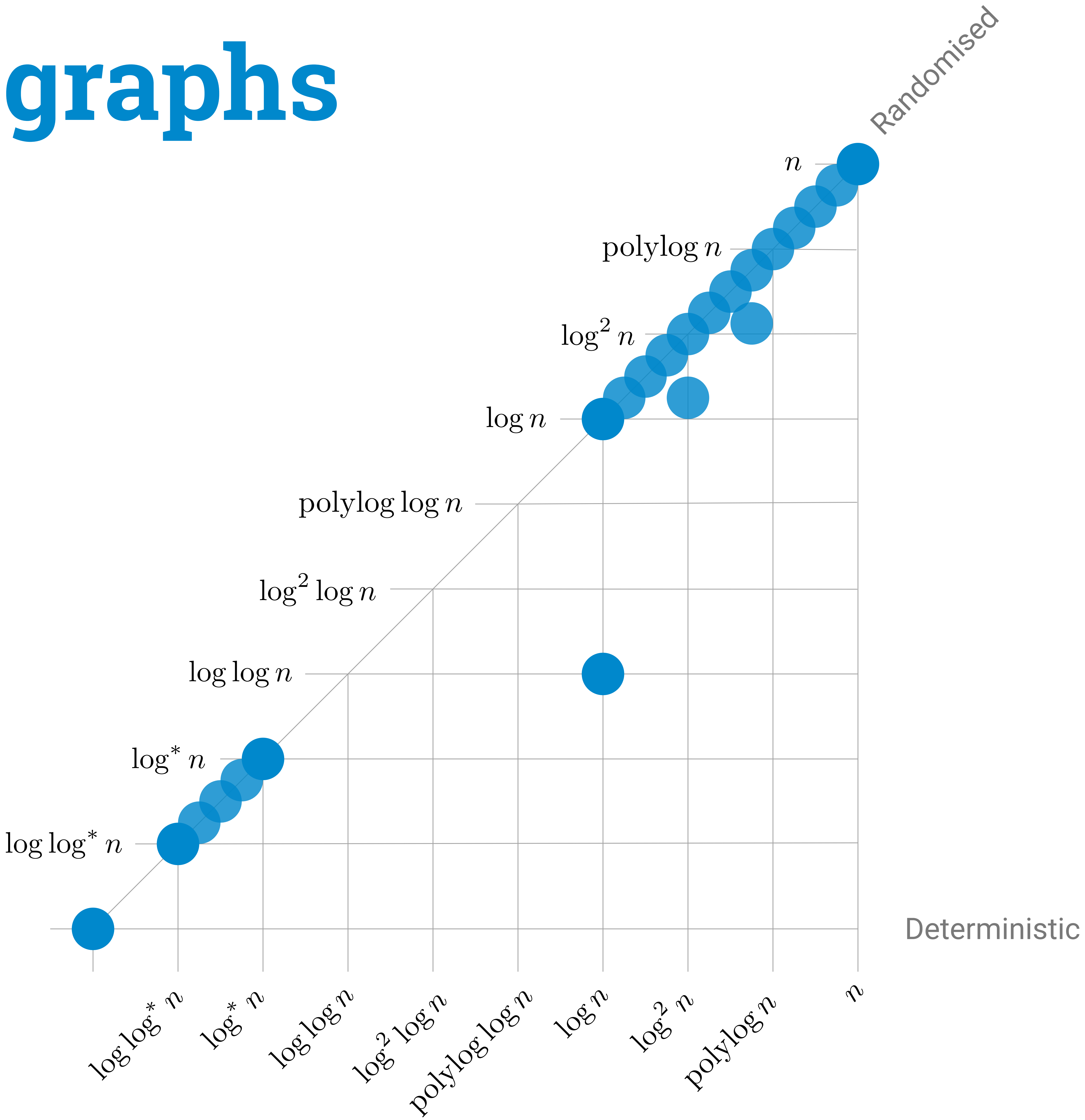

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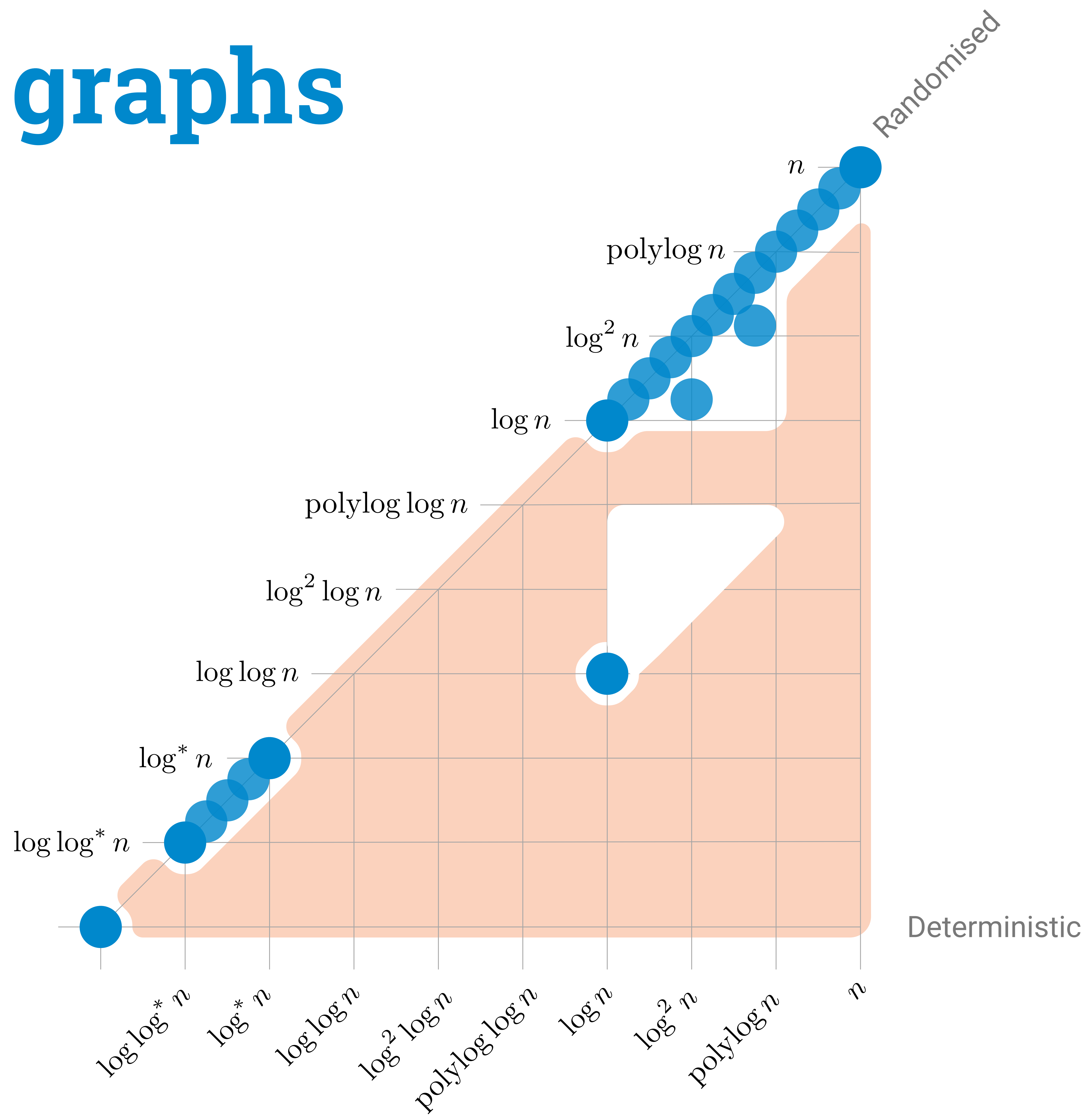

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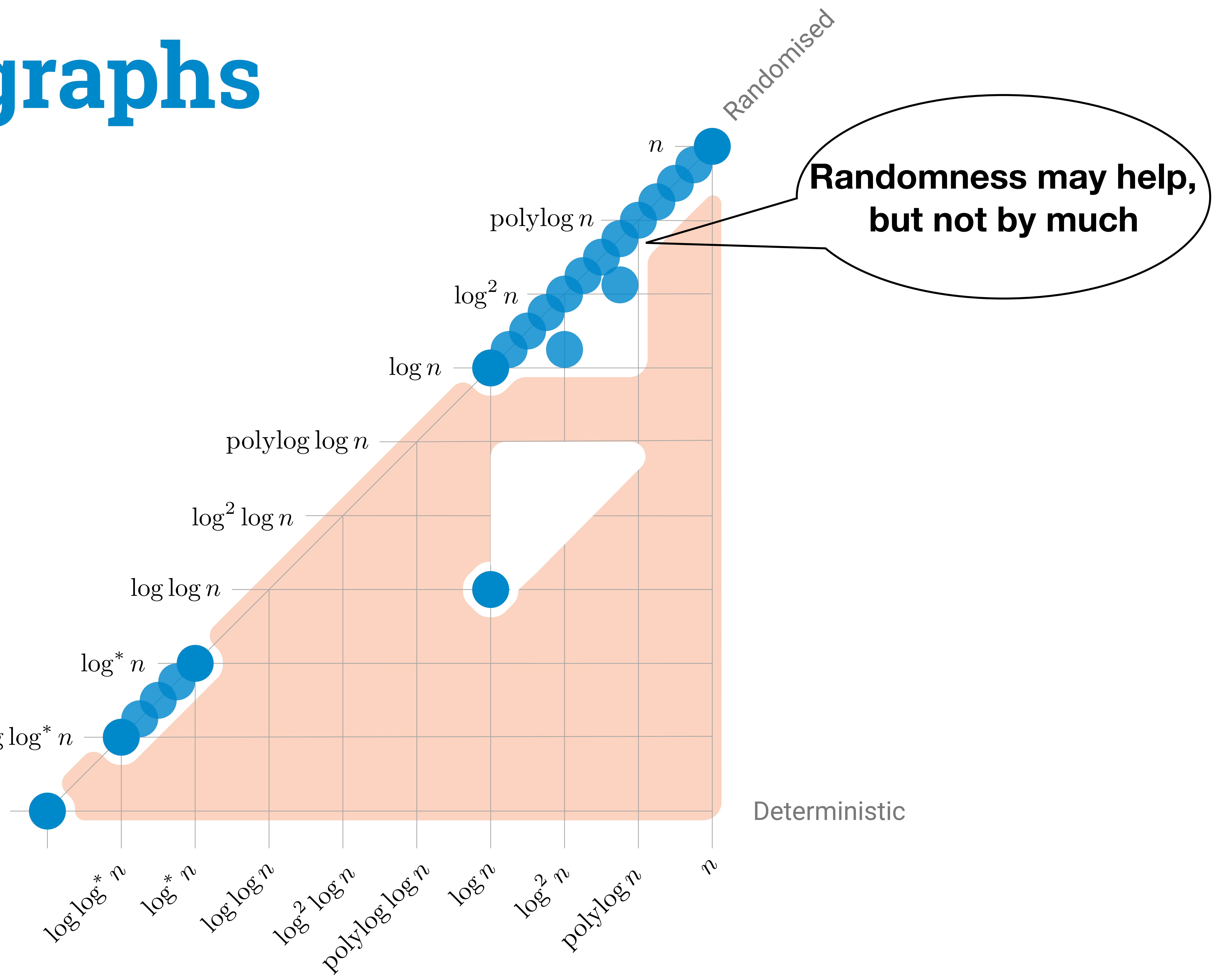

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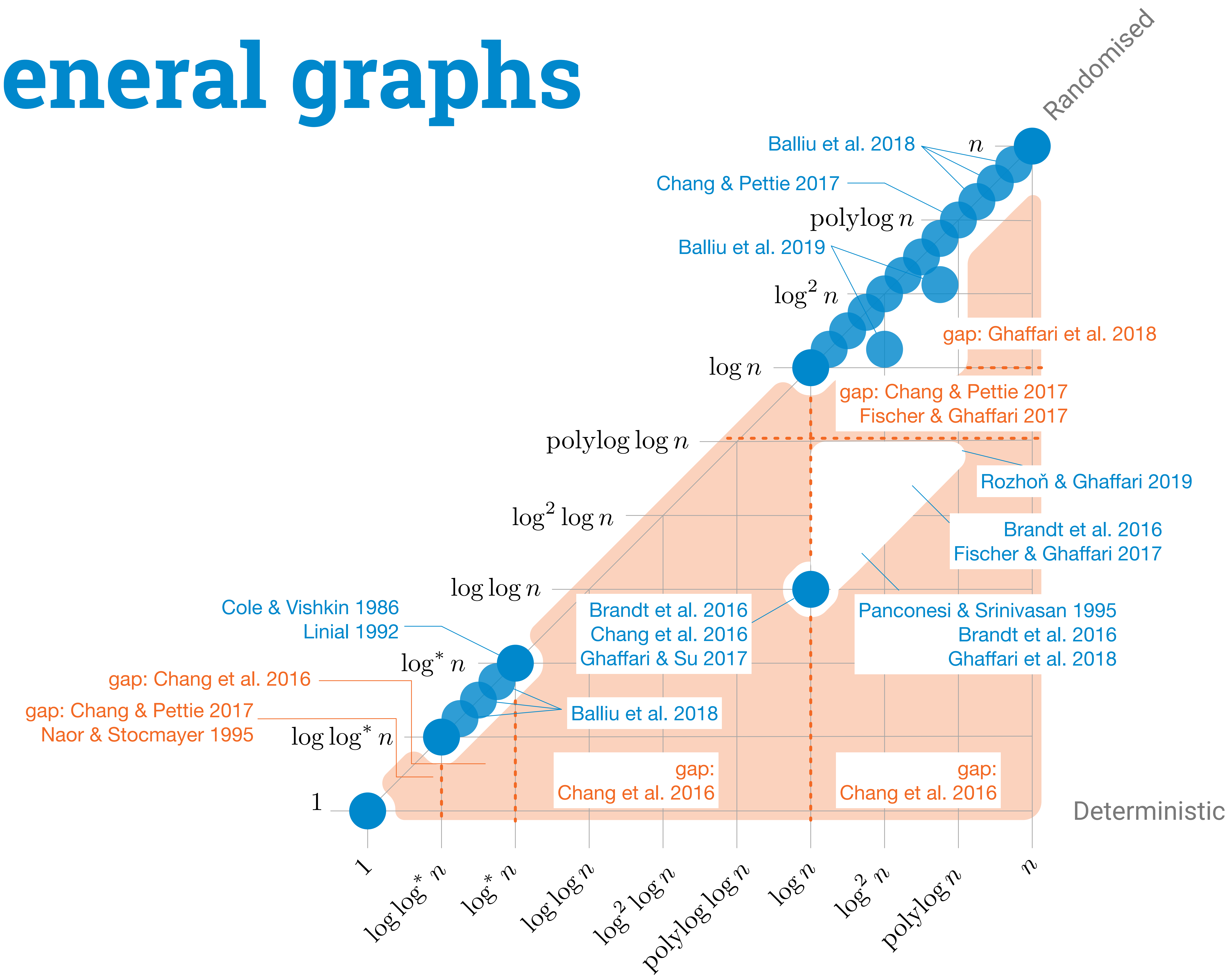

Artificial problems

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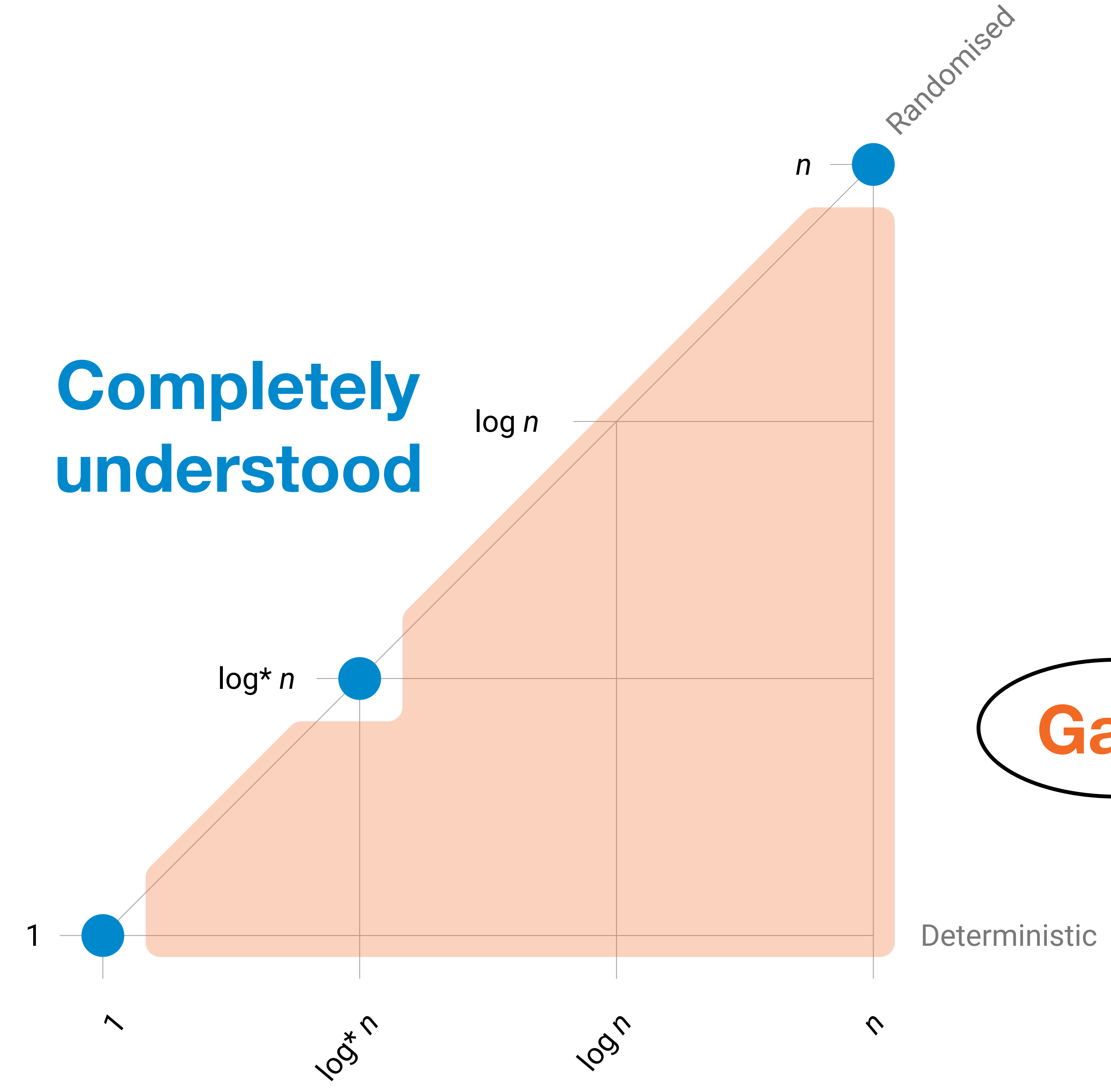
  - Prove that the diameter is $\omega(n^{3/5})$
- Challenge:
  - It should not be possible to prove that the diameter is too high when it is not
  - It must always be possible to prove that the diameter is too high if it is true, no matter what the graph is
  - The number of labels must be constant

Paths/Cycles



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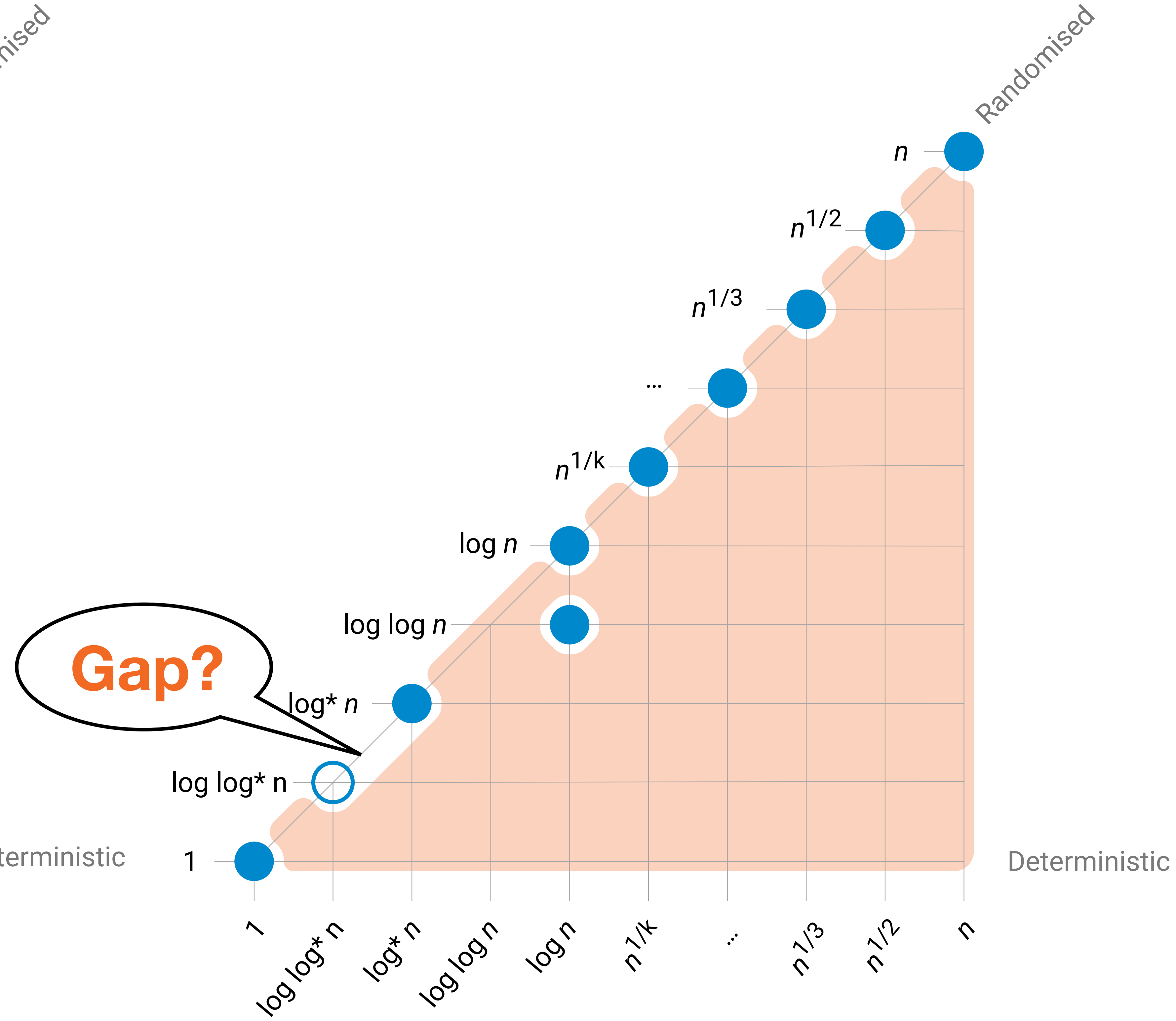

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