Distributed Property Testing

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Overview

- Decision Problems
- Property Testing (Centralized)
  - Dense model
  - Sparse model
- Distributed Property Testing

Decision problems

- Definition:
  - Given a property P
  - Given a graph G
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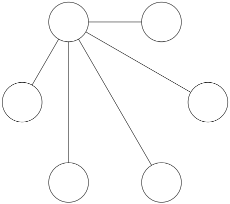
  - Is the graph k colorable?
- Often decision problems are hard
- Sometimes the input is huge, even linear time could be too much

Property testing

- Relax the requirements
- Given a property P
- Given a graph G
- Distinguish whether:
  - Does G satisfy the property P?
  - Is G far from satisfying the property P?
- The input is huge:
  - Only a small part of the input can be seen
  - We want sublinear algorithms

Example: 2 colorability



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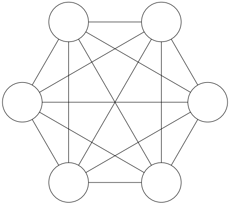
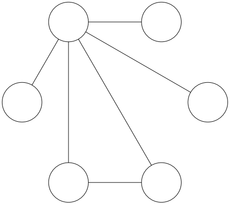


Far from being 2 colorable



Almost 2 colorable

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Let $G = (V, E)$, $n = |V|$, $m = |E|$. Let ϵ be a small constant in $(0, 1)$.
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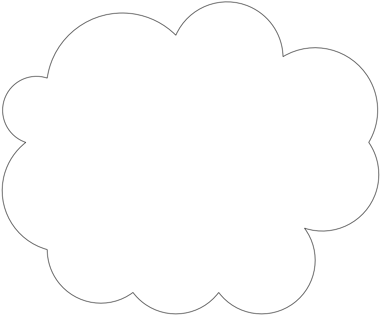
Sparse model

A graph is ϵ-far from satisfying a property if at least ϵm edges should be added or removed from G in order to make the property hold.

Complexity

- The complexity is measured in number of queries
- Different type of queries are allowed:



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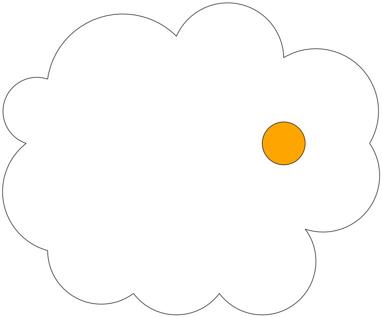

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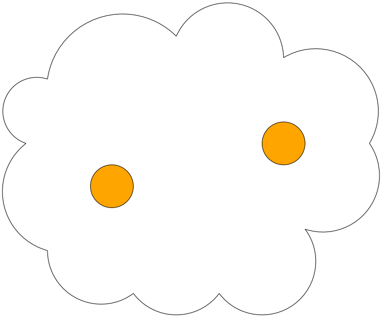

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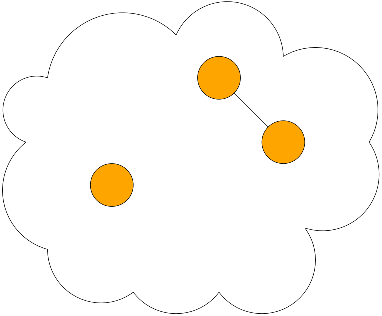

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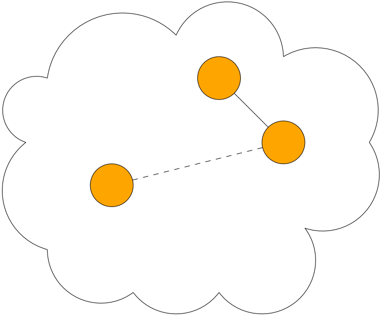
  - Give me a random neighbor of node x
  - Are nodes x and y neighbors?



Definition

Property Tester (2 sided error)

A tester for a graph property P is a randomized algorithm A that is required to accept or reject any given network instance, under the following two constraints:

- G satisfies $P \Rightarrow Pr[A \text{ accepts } G] \geq \frac{2}{3}$
- G is ϵ-far from satisfying $P \Rightarrow Pr[A \text{ rejects } G] \geq \frac{2}{3}$

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Subgraph freeness

We want to know if G does not contain any copy of a subgraph H, or if it contains many copies of H, being H some small graph (e.g. K_5).

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Graph removal lemma

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Lemma

H freeness can be tested in constant time, for any H of constant size.

A weaker lemma that holds in the sparse model

Lemma [Fraigniaud, Rapaport, Salo, Todinca '16]

Let H be any graph. Let G be an m-edge graph that is ϵ-far from being H-free. Then G contains at least $\epsilon m/|E(H)|$ edge-disjoint copies of H.

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The number of copies of H is proportional to m instead of $n^{|V(H)|}$, in the sparse model the problem is harder, in fact:

Lemma [Alon, Kaufman, Krivelevich, Ron '08]

Testing triangle freeness requires $\Omega(n^{\frac{1}{3}})$ queries.

Distributed property testing

Definition

A distributed tester for a graph property P is a distributed randomized algorithm A that satisfies the following conditions:

- G satisfies $P \Rightarrow$ every node outputs "accept"
- G is ϵ-far from satisfying $P \Rightarrow$ Pr[at least one node outputs "reject"] $\geq \frac{2}{3}$

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- All nodes start the computation at the same round
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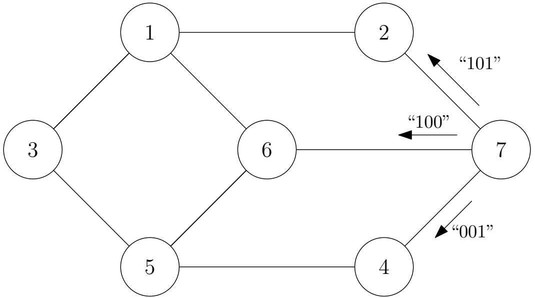

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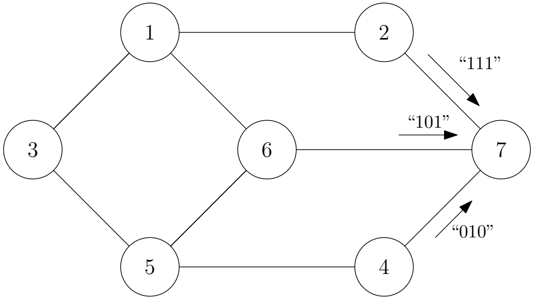
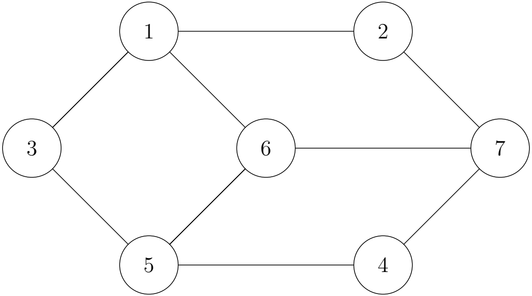


The Congest Model

- All nodes start the computation at the same round
- The computation proceeds in phases
- At each phase each node:
  - sends (possibly different) messages to its neighbors
  - receives messages sent by its neighbors
  - performs some local computation



The Congest Model

- The main constraint of the Congest model is that the exchanged messages should be small, typically $O(\log n)$.
- For example, messages of size $O(\log n)$ are enough to transmit, in a single round, a constant number of IDs of neighbors.

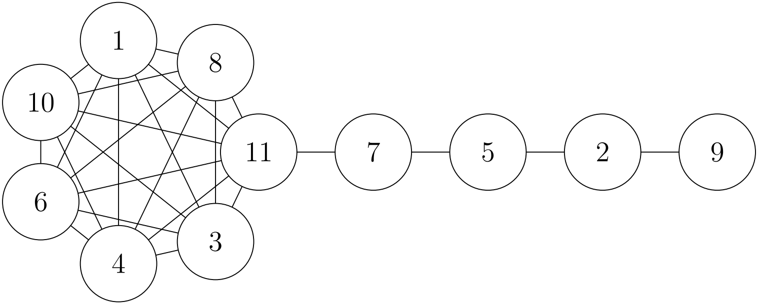
Decision problems in the Congest model

It is difficult to decide distributedly if a graph satisfies a property in constant time because of:

- Locality: two nodes can communicate in a time proportional to their distance
- Congestion: a node can not communicate all its neighbors to a single neighbor because they could be many

Example: knowing if a ring is 2 colorable requires to know the parity of its size.

Knowing the 2-hop neighborhood is hard



Dense model

Lemma [Censor-Hillel, Fischer, Schwartzman, Vasudev '16]

Any ϵ-tester for the dense model (for a non-disjointed property) that makes q queries can be converted to a distributed ϵ-tester that requires $O(q^2)$ rounds in the distributed setting.

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Example of properties testable in constant time:

- Is G H-free?
- Is G k-colorable?
- Is G a perfect graph?

Sparse model

[Censor-Hillel, Fischer, Schwartzman, Vasudev '16]

- Triangle freeness can be tested in $O(1/\epsilon^2)$
- Cycle freeness can be tested $O(\log n/\epsilon)$
- Cycle freeness requires at least $\Omega(\log n)$
- Bipartiteness can be tested in in $O(poly(\log \frac{n}{\epsilon}/\epsilon))$ in bounded degree graphs

[Fraigniaud, Rapaport, Salo, Todinca '16]

- H-freeness can be tested in constant time for any H s.t. $|V(H)| \leq 4$

Example: triangle freeness [Censor-Hillel, Fischer, Schwartzman, Vasudev '16]

Each node repeats $32\epsilon^{-2}$ times the following procedure:

- select two neighbors u and v uniformly at random
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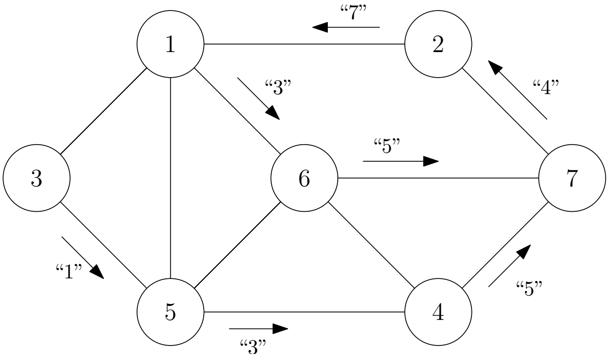

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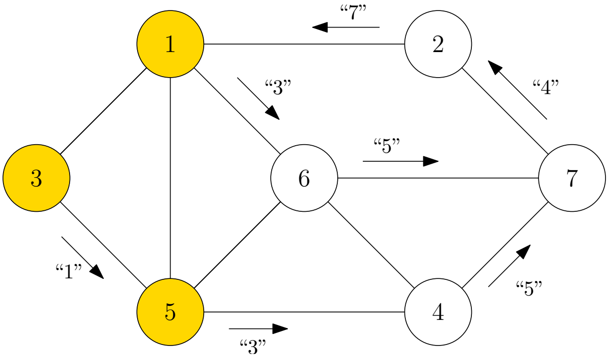

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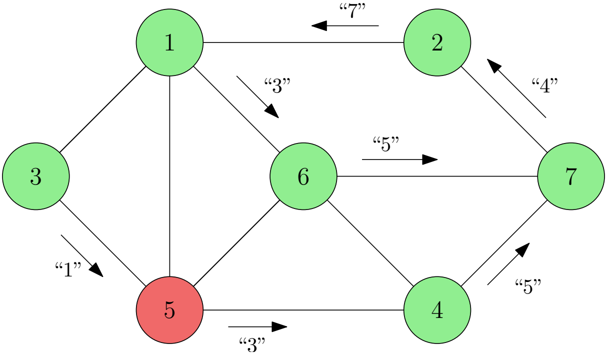

Generalization of the procedure: DFS and BFS testers

Each node:

- chooses some neighbor at random
- sends it to some random neighbor
- samples and propagates the received information

[Fraigniaud, Rapaport, Salo, Todinca '16]

BFS and DFS testers can not detect C_k and K_k for $k \geq 5$.

C_k detection

[Fraigniaud, O. '17]

There exists an ϵ-tester for C_k freeness, for any constant $k \geq 3$, that requires $O(\frac{1}{\epsilon})$ rounds in the CONGEST model.

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Procedure:

- Choose an edge u.a.r.
- Check if there is a cycle of length k passing through that edge
  - It can be done deterministically

Choose an edge at random

Lemma [Fraigniaud, Rapaport, Salo, Todinca '16]

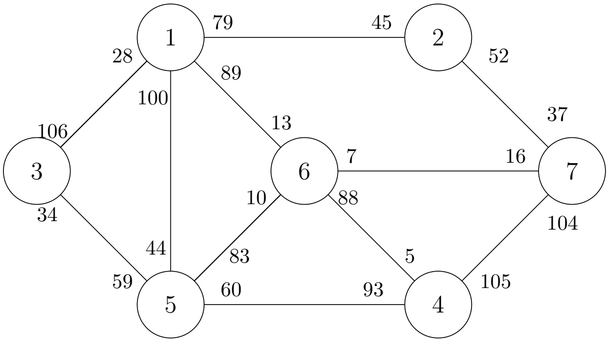
Let H be any graph. Let G be an m-edge graph that is ϵ-far from being H-free. Then G contains at least $\epsilon m/|E(H)|$ edge-disjoint copies of H.

This implies that by choosing a random edge we have probability $\Omega(\epsilon)$ to choose an edge that is part of some copy of H.

Choose an edge at random

- Each node picks a random weight w from $[1, m^2]$ for each edge incident to him
- The "leader" of each edge is the endpoint that chose the smaller weight
- If a node is the leader of multiple edges, choose the one with smaller weight
- Broadcast the edge of known minimum weight, and its weight, for a constant number of rounds

Choose an edge at random



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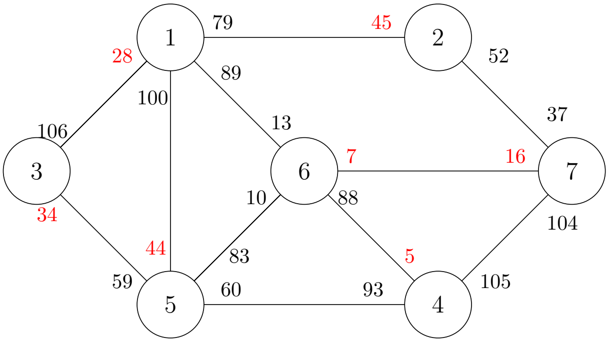

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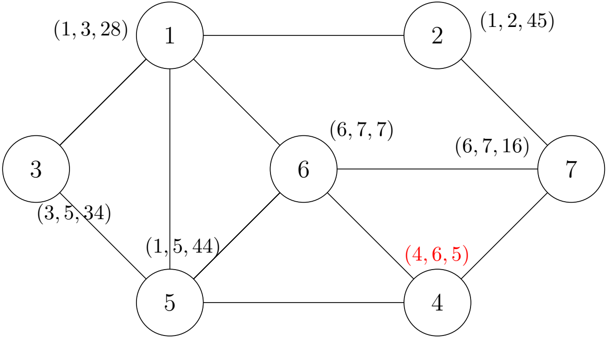

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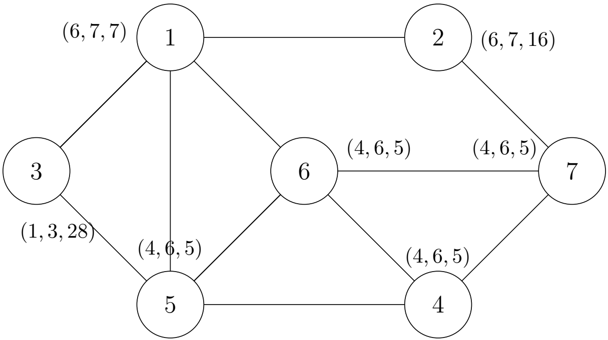

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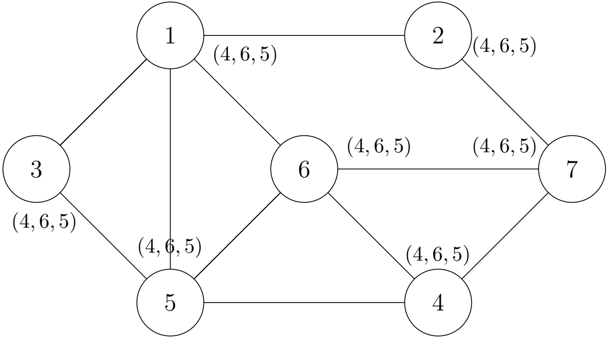
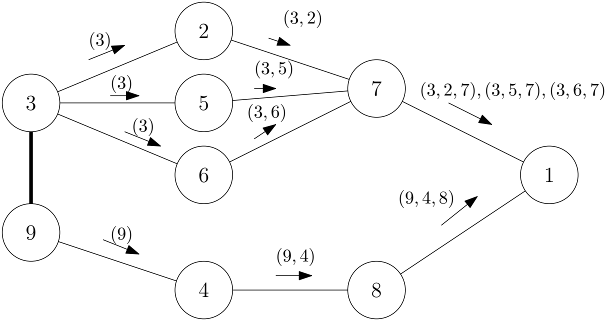


Check the presence of a cycle

Naïve solution:

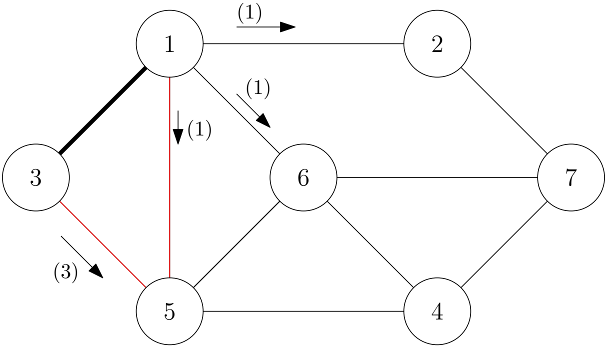
- The endpoints of the chosen edge broadcast their id
- Repeat
  - Append my id to each received sequence
  - Broadcast the new sequences just created
  - Check if two sequences are disjoint and form a cycle of desired length



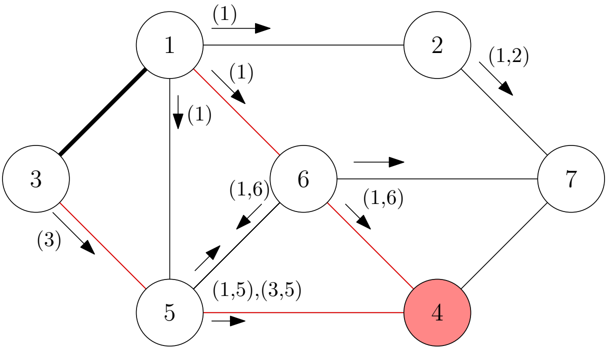
Example: triangle freeness

Repeat $O(\frac{1}{\epsilon})$ times:

- Choose an edge (u, v) as described before
- u and v broadcast
- If a node receives two messages a triangle is detected



C_5 detection

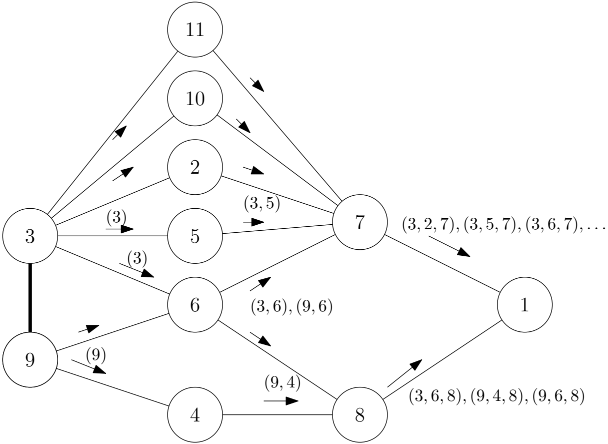


Node 4:

- receives $(1,5),(3,5),(1,6)$
- detects $(1,6,4,5,3)$

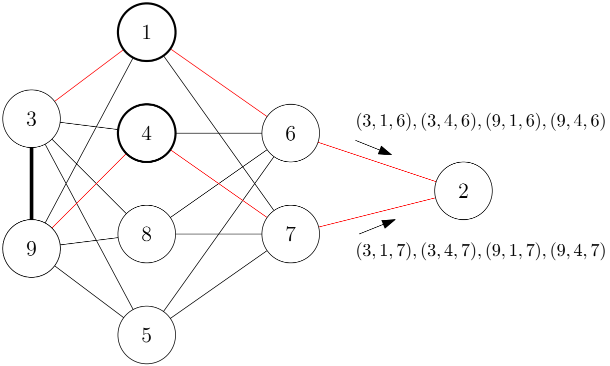
C_7 detection

- Nodes at distance 2 could potentially receive $\Theta(n)$ messages
- The previous procedure could require a lot of bandwidth



C_7 detection

- The partial solution can be sparsified
- For C_7, 3 subpaths (for each inital node) are enough



Sparsification of the intermediate solution

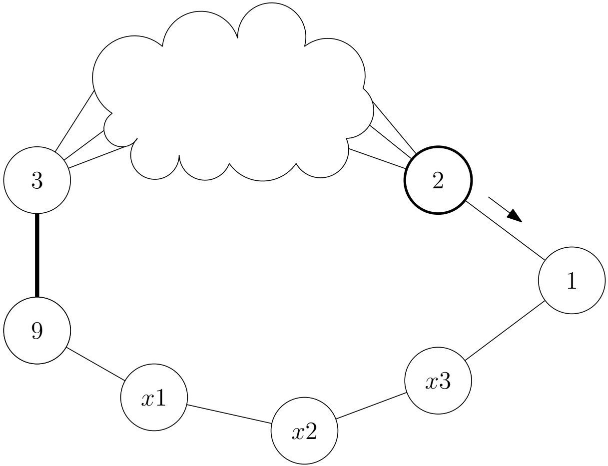
Lemma [Erdős, Hajnal, Moon '64]

Let V be a set of size n, and consider two integer parameters p and q. For any set $F \subseteq \mathcal{P}(V)$ of subsets of size at most p of V, there exists a *compact* (p, q)*-representation* of F, i.e., a subset $\hat{F}$ of F satisfying:

1. For each set $C \subseteq V$ of size at most q, if there is a set $L \in F$ such that $L \cap C = \emptyset$, then there also exists $\hat{L} \in \hat{F}$ such that $\hat{L} \cap C = \emptyset$;
2. The cardinality of $\hat{F}$ is at most $\binom{p+q}{p}$, for any $n \geq p + q$.

In other words, the number of subpaths that must be forwarded at each round do not depend on the size of the graph.

Sparsification of the intermediate solution



- Node 2 should send at least one sequence that does not contain $x1, x2$ and $x3$
- A constant number of sequences are enough

An easier (randomized) solution

- Pick a random edge (u, v)
- Each node picks a random color from $[1, k]$
- with constant probability the nodes of a cycle going from u to v will have colors $1, 2, \ldots, k$
- Start a BFS from u, that at round i can pass only on nodes with color i
- If the BFS reaches v at round k, a cycle is detected

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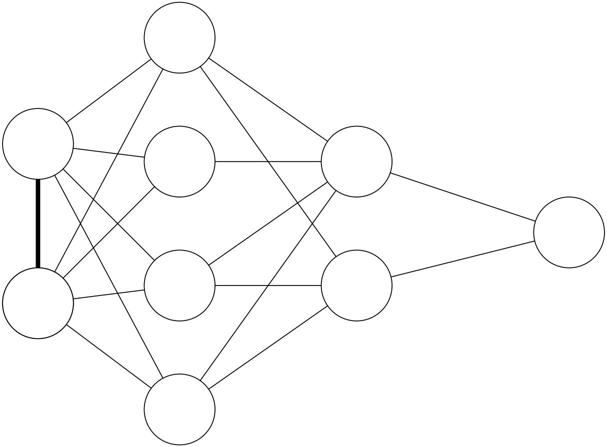

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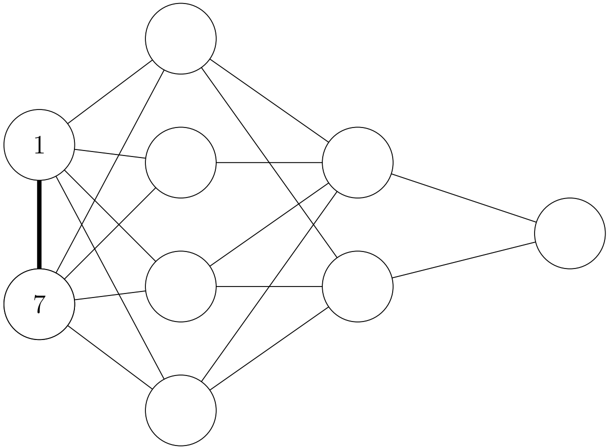

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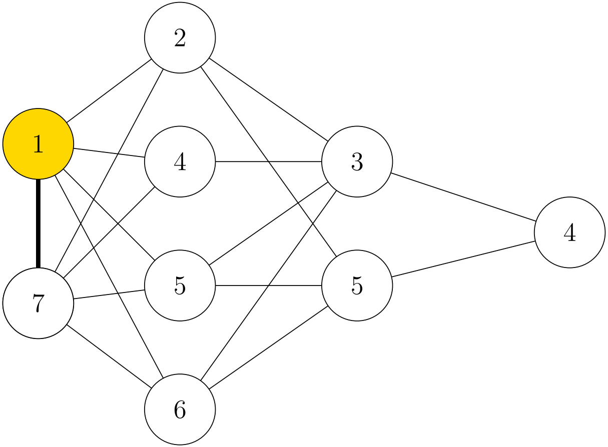

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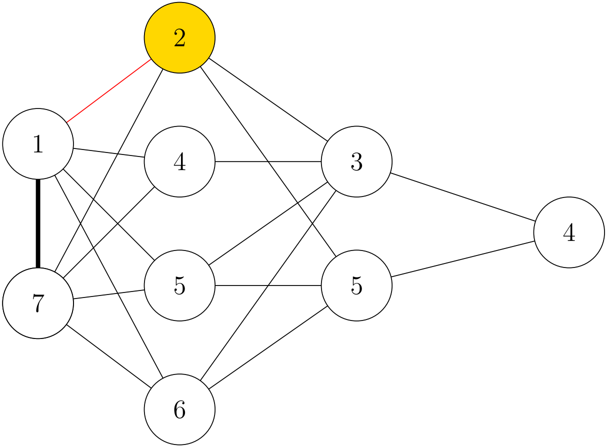

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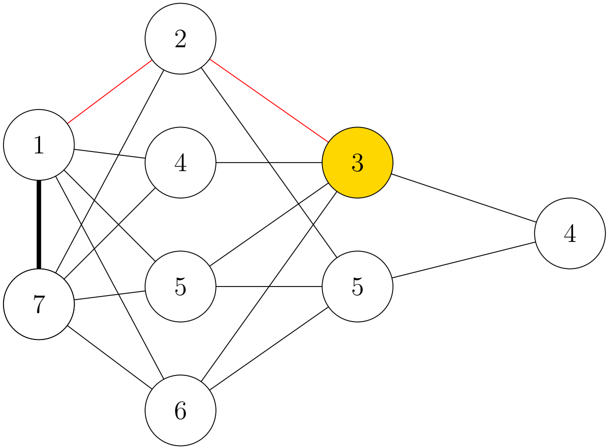

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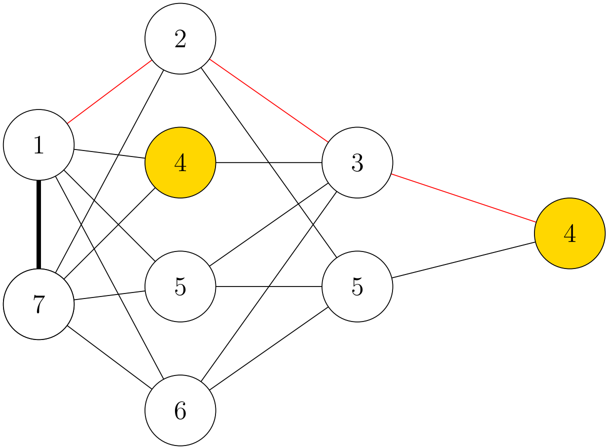

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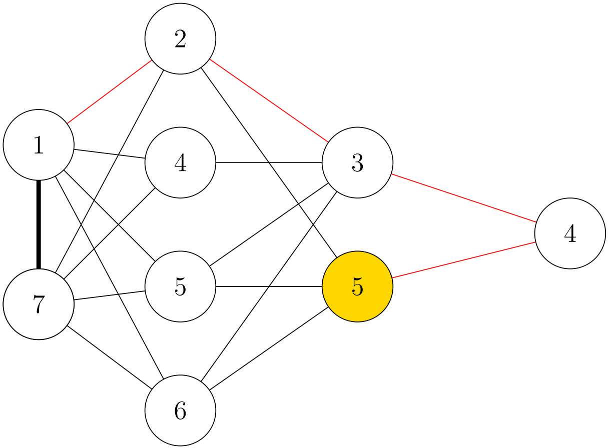

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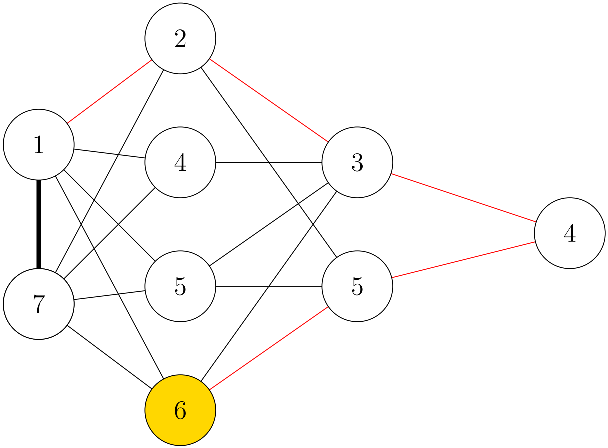

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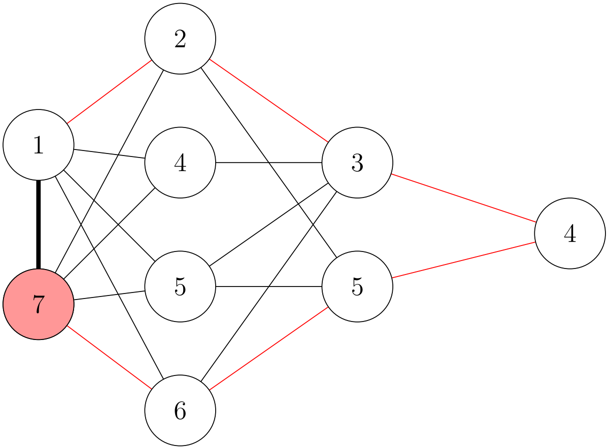
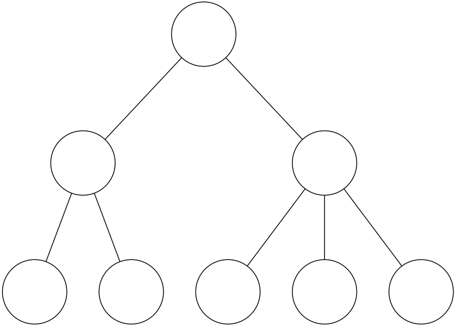


Tree detection

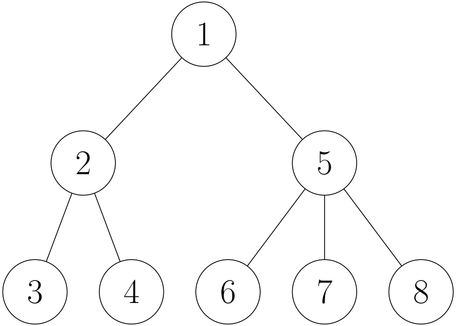
[Fraigniaud, Montealegre, O., Rapaport, Todinca '17]

In the CONGEST model, it is possible to check the presence of a fixed tree T of constant size, in $O(1)$ rounds, deterministically.

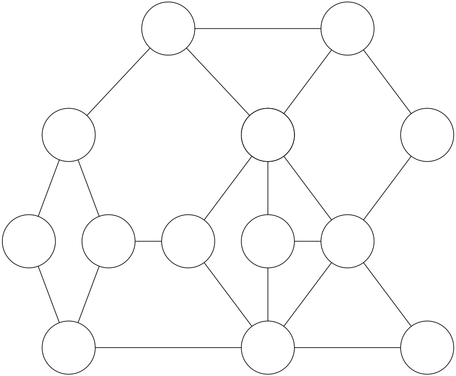
Tree detection: example

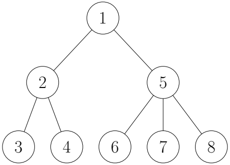


Tree detection: example

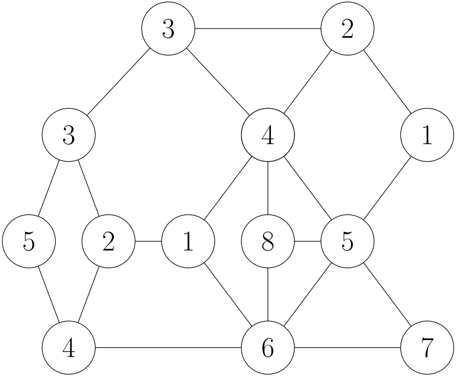


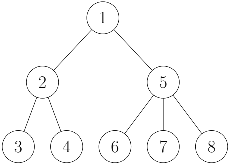
Tree detection: example



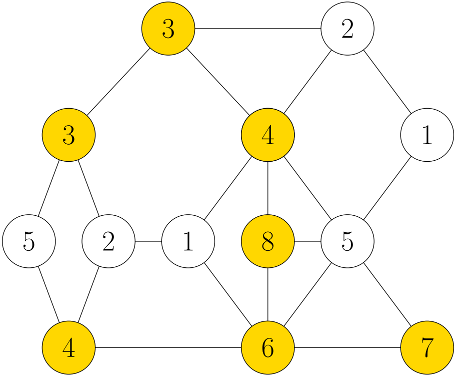


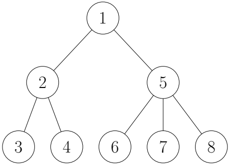
Tree detection: example



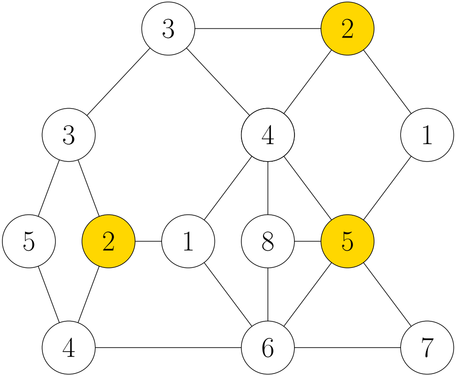


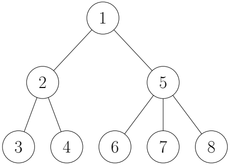
Tree detection: example



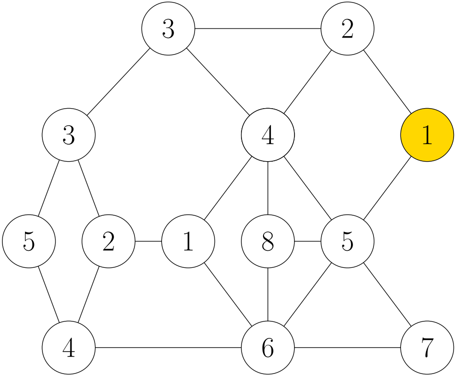


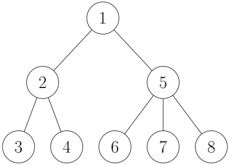
Tree detection: example



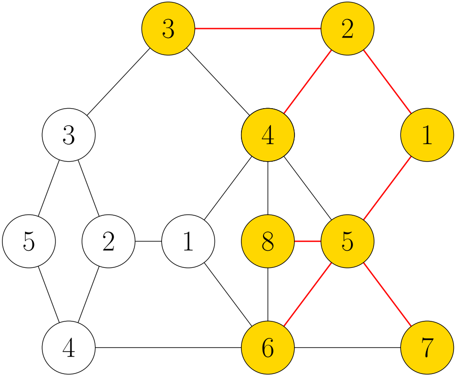


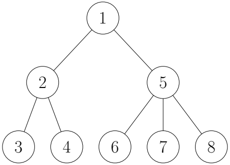
Tree detection: example



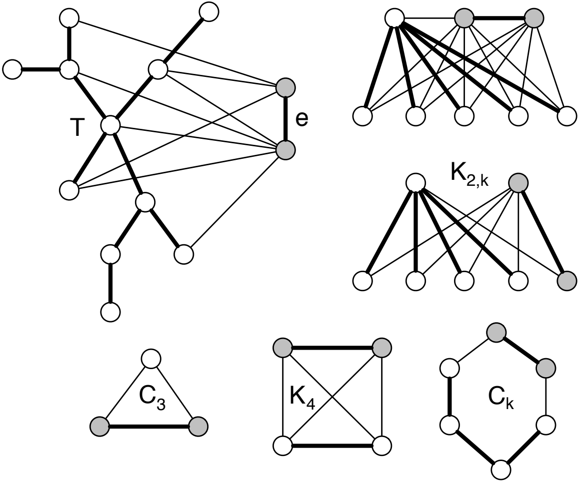


Tree detection: example





Tree + 1 edge

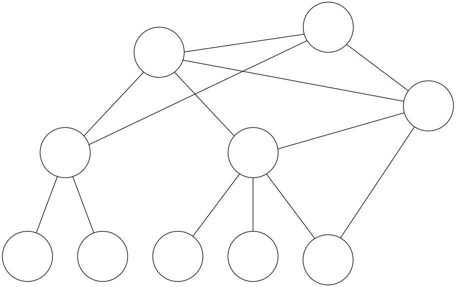


Tree + 1 edge

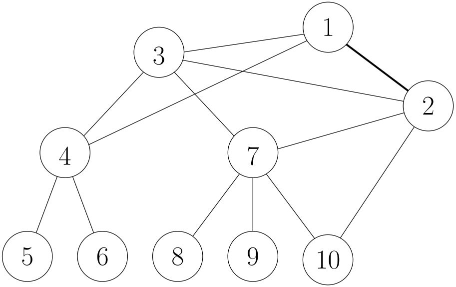
[Fraigniaud, Montealegre, O., Rapaport, Todinca '17]

There exists an ϵ-tester for H freeness, for any graph H of constant size composed by a tree, an edge, and arbitrary connections between the endpoints of the edge and the nodes of the tree, that requires $O(\frac{1}{\epsilon})$ rounds in the CONGEST model.

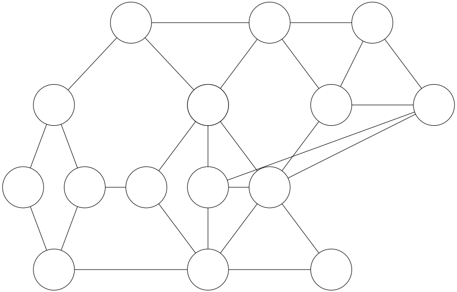
Tree + 1 edge: example

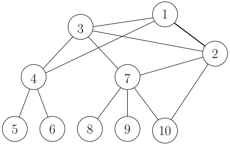


Tree + 1 edge: example

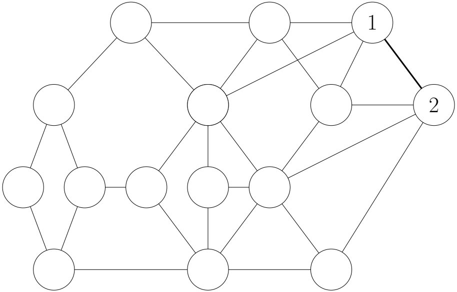


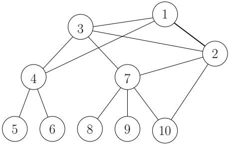
Tree + 1 edge: example





Tree + 1 edge: example





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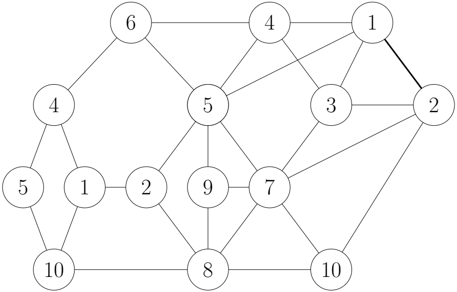
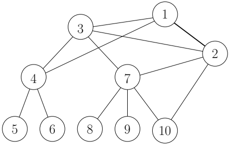
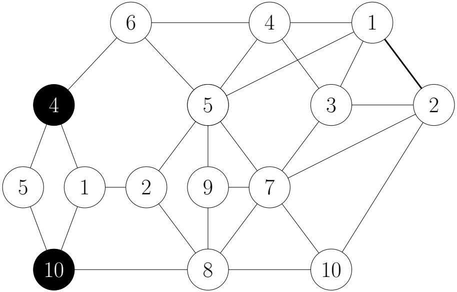
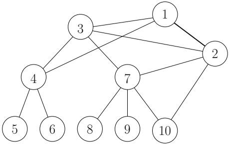




Tree + 1 edge: example





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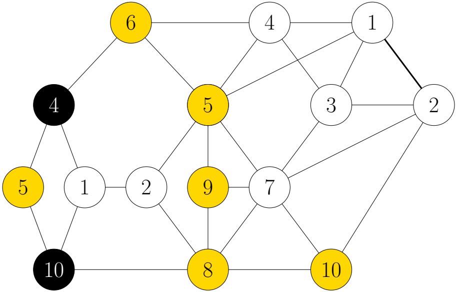
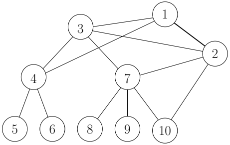
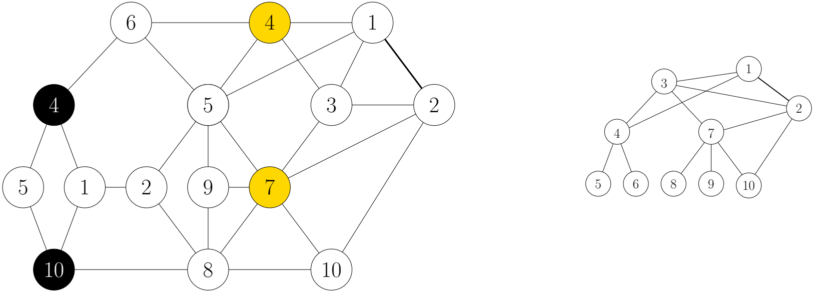
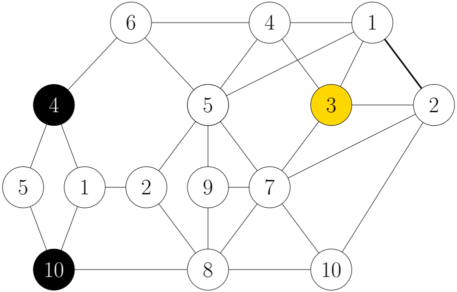


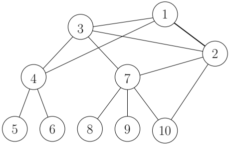


Tree + 1 edge: example

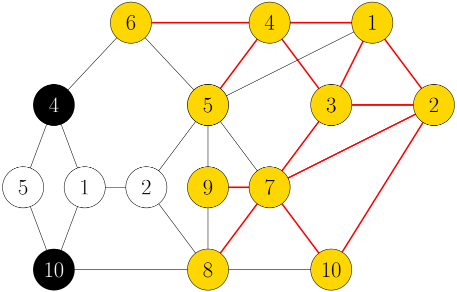


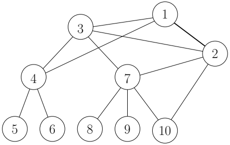
Tree + 1 edge: example



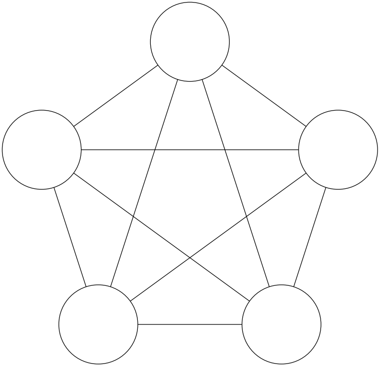


Tree + 1 edge: example

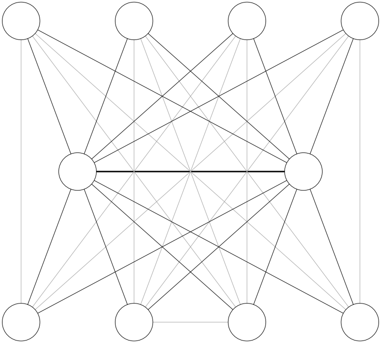




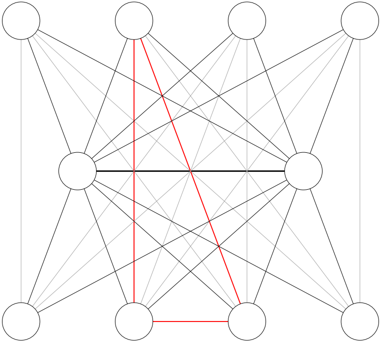
Open problems



Open problems



Open problems



Does there exist an ϵ-tester for K_5-freeness?

Conclusions

- There exists a deterministic algorithm for the CONGEST model that can check the presence of a fixed tree in a constant number of rounds
- There exists an ϵ-tester for the CONGEST model that can check the presence of a fixed tree + 1 edge in $O(1/\epsilon)$
- The minimal graph not testable with the above algorithms is K_5
- For distributed property testing, no lower bounds are known!

Thank you